THE

MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

ROYAL NAVAL COLLEGE, GREENWICH, LONDON, S.E. 10.

LONDON

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

VOL. XXXII DECEMBER, 1948 No. 302

'SALMON'.

THE learner of one generation is the teacher of the next, and as a rule it is only for a decade or so that a textbook or treatise plays its part in the development of its subject directly ; what is original in matter or method is absorbed by readers and permeates the books which they in their turn write. Chrystal's *Algebra* and Hobson's *Trigonometry* are fountain-heads, but the young mathematician does not go to them now ; he imbibes their wisdom through channels fashioned in his own lifetime.

To the rule there is one outrageous incredible glorious exception : 'Salmon'. The name stands not for an unknown provost and professor of divinity, not generically for the four mathematical treatises of which three have suffered the common fate. 'Salmon' is 'A Treatise on Conic Sections containing an account of the most important modern algebraic and geometric methods' and nothing else. And our frontispiece shows that this book, which schoolboys and undergraduates of to-day are urged to read as a matter of course, has in sober fact reached its centenary. It is the moment for our salute.

E. H. N.

The thanks of the Association are due to the University Library, Edinburgh, for permission to photograph their copy.

PONCELET'S PORISTIC POLYGONS.

By J. A. Todd.

1. Poncelet's well-known theorem states that if two conics S and S' in a plane are such that an n-gon exists whose vertices lie on S' and whose sides touch S, then an infinite number of such polygons exist. When this is the case, a relation holds between the invariants of the two conics. The explicit determination of this relation is due to Cayley,* whose result is quoted by Salmon in a footnote on p. 342 of his *Conic Sections* (reprint of 1929). With a slight change of notation, Cayley's result takes the following form.

Let

$$\Delta(\lambda) \equiv \Delta + \Theta\lambda + \Theta'\lambda^2 + \Delta'\lambda^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

be the discriminant of the conic $S + \lambda S' = 0$, so that Δ, Θ, Θ', Δ' are the invariants of the two conics S and S', and suppose that the expansion of $\sqrt{\Delta(\lambda)}$ as a power series in ascending powers of λ is

$$\sqrt{\Delta(\lambda)} \equiv C(\lambda) = \sum_{r=0}^{\infty} c_r \lambda^r. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Then, if we write

$$u_{2k} = \begin{vmatrix} c_2 & c_3 & \cdots & c_{k+1} \\ c_3 & c_4 & \cdots & c_{k+2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{k+1} & c_{k+2} & \cdots & c_{2k} \end{vmatrix}, \quad u_{2k+1} = \begin{vmatrix} c_3 & c_4 & \cdots & c_{k+2} \\ c_4 & c_5 & \cdots & c_{k+3} \\ \cdots & \cdots & \cdots & \cdots \\ c_{k+2} & c_{k+3} & \cdots & c_{2k+1} \end{vmatrix}, \ldots\ldots(3)$$

the condition that there exist n-gons inscribed in S' and circumscribed to S is

$$u_{n-1} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Cayley proved this by consideration of properties of elliptic functions. This approach is, in fact, the most natural method of attack, and has formed the basis for most subsequent work on the subject. But the simple form taken by Cayley's result presents a challenge to produce an elementary analytical proof, assuming only what is familiar in the standard accounts of the theory of invariants of a pair of conics (such as Salmon's own account, which is one of the best). Such a proof is presented below. Any such proof is liable, of course, to present close contacts with Cayley's own (in a suitably disguised form), and I make no pretence that what follows is a really "independent" treatment. On the contrary, the device of § 4 below, which is the critical step in the argument, was suggested directly by the delightful geometrical account given by Lebesgue,† whose paper consists essentially in replacing Cayley's transcendental arguments by geometrical considerations of residuation on a plane cubic curve. The sole aim of this paper is to make a proof of Cayley's result available to the reader familiar only with the ideas and technique of Salmon; and since the purpose of the present account is expository, I have not hesitated, in order to make the paper self-contained, to reproduce some fairly familiar work in §§ 2 and 3.

2. We start by proving the following lemma,‡ which is fundamental.

If one side of a triangle inscribed in a conic S' touches a conic S, and if a second side touches a conic $S + qS'$ of the pencil determined by S and S', then the envelope of the third side consists of the two conics $S + r_1S'$, $S + r_2S'$ of the pencil, where r_1 and r_2 are the roots of the quadratic in r.

$$\Delta'^2q^2r^2 - 2\Delta'(2\Delta + \Theta q)r + [(\Theta^2 - 4\Delta\Theta') - 4\Delta\Delta'q] = 0. \quad \ldots\ldots\ldots\ldots(5)$$

* Cayley, *Phil. Mag.* (4), 6 (1853), 99; *Collected Papers*, II, 87.
† Lebesgue, *Ann. Fac. Sci. Toulouse* (3), 13 (1922), 61.
‡ *cf.* Salmon, *loc. cit.*, p. 343.

Let XYZ be a general triangle satisfying the conditions, so that YZ touches S at P and ZX touches $S+qS'$ at Q. Let XP, YQ meet in E and let ZE meet XY in R. Then, if XYZ is taken as the triangle of reference and E as the unit point, the coordinates of P, Q and R are respectively (0, 1, 1), (1, 0, 1), (1, 1, 0). Since S' passes through XYZ, its equation is of the form

$$S' \equiv 2(fyz+gzx+hxy)=0, \qquad \text{.............................(6)}$$

and, since S touches YZ at P and $S+qS'$ touches ZX at Q, the equation of S is of the form

$$S \equiv x^2+y^2+z^2-2yz-2(1+gq)zx-2(1+hr)xy=0 \qquad \text{....................(7)}$$

for some value of r. From (6) and (7) we see that $S+rS'$ touches XY at R. We now show that q and r satisfy (5), from which (since the coefficients in this equation are invariants) our result follows at once. A simple calculation shows that the invariants of S and S' are given by

$$\Delta = -(2+gq+hr)^2, \quad \Theta = 2(f+g+h)(2+gq+hr)+2fghqr,$$
$$\Theta' = -(f+g+h)^2-2fgh(q+r), \quad \Delta' = 2fgh,$$

from which, eliminating f, g, h, we obtain

$$(\Theta-\Delta' qr)^2 = 4\Delta[\Theta'+\Delta'(q+r)], \qquad \text{..........................(8)}$$

which is clearly equivalent to (5). We observe that this relation is symmetrical in q and r.

3. We can now prove the following theorem:

If $A_1A_2 \ldots A_n$ is an n-gon inscribed in a conic S', and if A_1A_2, A_2A_3, ..., $A_{n-1}A_n$ all touch a conic S, then A_1A_n touches the conic $S+\lambda_n S'$, where

$$\lambda_3 = \frac{\Theta^2-4\Delta\Theta'}{4\Delta\Delta'}, \quad \lambda_4 = \frac{2(2\Delta+\Theta\lambda_3)}{\Delta'\lambda_3^2}, \qquad \text{........................(9)}$$

and, if $n>4$,

$$\lambda_n\lambda_{n-2} = \frac{(\Theta^2-4\Delta\Theta')-4\Delta\Delta'\lambda_{n-1}}{\Delta'^2\lambda_{n-1}^2}, \quad \lambda_n+\lambda_{n-2} = \frac{2(2\Delta+\Theta\lambda_{n-1})}{\Delta'\lambda_{n-1}^2}. \qquad \text{......(10)}$$

It is to be understood, in this enunciation, that a pair of consecutive sides $A_{r-1}A_r$, A_rA_{r+1} of the polygon are distinct unless, exceptionally, A_r is a common point of S and S'.

This follows very simply from the lemma of § 2. In fact, when $n=3$, we put $q=0$ in (5) which then reduces to a linear equation whose root is the value of λ_3 defined in (9). Hence the third side of a triangle inscribed in S' and having two sides tangent to S is the conic $S+\lambda_3 S'$. Suppose, then, that $n=4$. Then, if $A_1A_2A_3A_4$ is a quadrangle inscribed in S' such that A_1A_2, A_2A_3, A_3A_4 touch S, A_1A_3 touches $S+\lambda_3 S'$. By applying the lemma of § 2 to the triangle $A_1A_3A_4$, where A_1A_3 touches $S+\lambda_3 S'$ and A_3A_4 touches S, we see that A_1A_4 touches one of the two conics $S+rS'$ where r satisfies (5) with $q=\lambda_3$. One root of this equation is zero, and corresponds to the fact that A_3A_2 is one of the tangents to S from A_3 while A_1A_2 touches S. The other root is clearly λ_4 as defined by (9), and $S+\lambda_4 S'$ is thus the envelope of A_1A_4.

For $n>4$ we proceed inductively. Assuming that A_1A_{n-1} touches $S+\lambda_{n-1}S'$ we see, by considering the triangles $A_1A_{n-1}A_n$ and $A_1A_{n-1}A_{n-2}$, that A_1A_n envelops a conic $S+\lambda_n S'$ of the pencil, and that λ_n and λ_{n-2} are the roots of (5) when $q=\lambda_{n-1}$. Whence, by the formula for the product and sum of the roots of a quadratic equation, the relations (10) follow.

4. Up to this point our argument has been simple and fairly familiar. The difficulties arise when we attempt to obtain an explicit expression for λ_n from the recurrence relations (10). Actually it will appear that, for $k \geqslant 1$,

$$\lambda_{2k+1} = -\frac{2\Delta^{\frac{1}{2}}}{\Delta'} \cdot \frac{u_{2k-2}u_{2k}}{u_{2k-1}^2}, \quad \lambda_{2k+2} = 2\Delta^{\frac{1}{2}} \cdot \frac{u_{2k-1}u_{2k+1}}{u_{2k}^2}, \quad \text{..........(11)}$$

where the u's are the functions defined in (3) with the conventional addition that $u_0 = u_1 = 1$, and where the sign of $\Delta^{\frac{1}{2}}$ is chosen so that, in (2), $c_0 = +\Delta^{\frac{1}{2}}$. (It is easily verified by direct calculation that, with these conventions, the values of λ_3 and λ_4 are given by (9)).

But I see no simple direct way of verifying that the expressions for λ_n defined by (11) satisfy the recurrence relations (10). We shall therefore proceed indirectly, proving first of all the following theorem.

If λ_n is defined by (9) *and* (10) *then, for any $n \geqslant 3$, there exist polynomials $P_n(\lambda)$, $Q_n(\lambda)$, free from common factors, such that*

$$[P_n(\lambda)]^2 - 4\Delta\Delta(\lambda)[Q_n(\lambda)]^2 \equiv d_n\lambda^{n-1}(\lambda - \lambda_n)\,; \quad \text{......................(12)}$$

where $\Delta(\lambda)$ is defined by (1) *and d_n is a constant, and where the degrees of $P_n(\lambda)$ and $Q_n(\lambda)$ may be taken not to exceed, respectively, k and $k-1$ if $n = 2k+1$, or $k+1$ and $k-1$ if $n = 2k+2$.*

The proof of this result consists of a verification for $n = 3$, 4 followed by a process of induction. It rests essentially on two identities. The first of these is

$$[(\Theta - \Delta' qr)\lambda + 2\Delta]^2 - 4\Delta\Delta(\lambda) \equiv -4\Delta\Delta'\lambda(\lambda - q)(\lambda - r), \quad \text{.............(13)}$$

if q, r are connected by (5) or the equivalent (8); and is verified at once by equating coefficients. The second is the simple identity

$$(P^2 - 4\Delta\Delta(\lambda)Q^2)(R^2 - 4\Delta\Delta(\lambda)S^2) \equiv (PR + 4\Delta\Delta(\lambda)QS)^2 - 4\Delta\Delta(\lambda)(PS + QR)^2, \quad (14)$$

which shows that the product of two expressions of the form $P^2 - 4\Delta\Delta(\lambda)Q^2$ can be expressed rationally in the same form.

We turn now to the proof of the theorem.

(i) *The case* $n = 3$.

Since (8) holds with $q = 0$, $r = \lambda_3$, we have, by (13),

$$(\Theta\lambda + 2\Delta)^2 - 4\Delta\Delta(\lambda) \equiv -4\Delta\Delta'\lambda^2(\lambda - \lambda_3), \quad \text{..................(15)}$$

which is of the required form with

$$P_3(\lambda) \equiv \Theta\lambda + 2\Delta, \quad Q_3(\lambda) = 1, \quad d_3 = -4\Delta\Delta'. \quad \text{.................(16)}$$

(ii) *The case* $n = 4$.

We shall find it convenient to write

$$R_n(\lambda) \equiv (\Theta - \Delta'\lambda_{n-1}\lambda_n)\lambda + 2\Delta. \quad \text{..........................(17)}$$

Since (8) holds with $q = \lambda_3$, $r = \lambda_4$, we have, by (13),

$$[R_4(\lambda)]^2 - 4\Delta\Delta(\lambda) \equiv -4\Delta\Delta'\lambda(\lambda - \lambda_3)(\lambda - \lambda_4). \quad \text{.................(18)}$$

We form the product of the left-hand members of (15) and (18), and use (14) with $P = P_3(\lambda)$, $Q = Q_3(\lambda)$, $R = R_4(\lambda)$, $S = 1$. We then obtain

$$[X_4(\lambda)]^2 - 4\Delta\Delta(\lambda)[Y_4(\lambda)]^2 \equiv 16\Delta^2\Delta'^2\lambda^3(\lambda - \lambda_3)^2(\lambda - \lambda_4), \quad \text{..........(19)}$$

where
$$X_4(\lambda) = P_3(\lambda)R_4(\lambda) + 4\Delta\Delta(\lambda)Q_3(\lambda), \quad \text{.......................(20)}$$

$$Y_4(\lambda) = P_3(\lambda) + Q_3(\lambda)R_4(\lambda). \quad \text{..............................(21}$$

We proceed to prove that $\lambda - \lambda_3$ is a factor of $X_4(\lambda)$ and $Y_4(\lambda)$. In fact, from (16), (17) and (9) we have

$$\begin{aligned}
X_4(\lambda) &= (\Theta\lambda + 2\Delta)[(\Theta - \Delta'\lambda_3\lambda_4)\lambda + 2\Delta] + 4\Delta(\Delta'\lambda^3 + \Theta'\lambda^2 + \Theta\lambda + \Delta) \\
&= 4\Delta\Delta'\lambda^3 + (4\Delta\Theta' + \Theta^2 - \Theta\Delta'\lambda_3\lambda_4)\lambda^2 + (8\Delta\Theta - 2\Delta\Delta'\lambda_3\lambda_4)\lambda + 8\Delta^2 \\
&= 4\Delta\Delta'\lambda^3 + \left[4\Delta\Theta' + \Theta^2 - \Theta\left(2\Theta + \frac{4\Delta}{\lambda_3}\right)\right]\lambda^2 + \left[8\Delta\Theta - 2\Delta\left(2\Theta + \frac{4\Delta}{\lambda_3}\right)\right]\lambda + 8\Delta^2 \\
&= 4\Delta\Delta'\lambda^3 + \left[-4\Delta\Delta'\lambda_3 - \frac{4\Delta\Theta}{\lambda_3}\right]\lambda^2 + \left[4\Delta\Theta - \frac{8\Delta^2}{\lambda_3}\right]\lambda + 8\Delta^2 \\
&= 4\Delta\Delta'\lambda^2(\lambda - \lambda_3) + (4\Delta\Theta\lambda + 8\Delta^2)\left(1 - \frac{\lambda}{\lambda_3}\right) \\
&= \frac{4\Delta}{\lambda_3}(\Delta'\lambda_3\lambda^2 - \Theta\lambda - 2\Delta)(\lambda - \lambda_3),
\end{aligned}$$

$$\begin{aligned}
Y_4(\lambda) &= (\Theta\lambda + 2\Delta) + [(\Theta - \Delta'\lambda_3\lambda_4)\lambda + 2\Delta] = (2\Theta - \Delta'\lambda_3\lambda_4)\lambda + 4\Delta \\
&= \left[2\Theta - \left(2\Theta + \frac{4\Delta}{\lambda_3}\right)\right]\lambda + 4\Delta = 4\Delta\left(1 - \frac{\lambda}{\lambda_3}\right) \\
&= -\frac{4\Delta}{\lambda_3}(\lambda - \lambda_3).
\end{aligned}$$

Thus, writing

$$X_4(\lambda) \equiv 4\Delta(\lambda - \lambda_3)P_4(\lambda), \quad Y_4(\lambda) \equiv 4\Delta(\lambda - \lambda_3)Q_4(\lambda), \quad \text{.............(22)}$$

we have, from (19),

$$[P_4(\lambda)]^2 - 4\Delta\Delta(\lambda)[Q_4(\lambda)]^2 = d_4\lambda^3(\lambda - \lambda_4), \quad \text{..................(23)}$$

where

$$P_4(\lambda) = \frac{1}{\lambda_3}(\Delta'\lambda_3\lambda^2 - \Theta\lambda - 2\Delta), \quad Q_4(\lambda) = -\frac{1}{\lambda_3}, \quad d_4 = \Delta'^2, \quad \text{..........(24)}$$

which proves our theorem for $n = 4$.

(iii) *The general case.*

Assuming our theorem true for suffix $n - 1$ or less, we define polynomials $X_n(\lambda)$, $Y_n(\lambda)$ by the relations

$$X_n(\lambda) = P_{n-1}(\lambda)R_n(\lambda) + 4\Delta\Delta(\lambda)Q_{n-1}(\lambda), \quad \text{....................(25)}$$

$$Y_n(\lambda) = P_{n-1}(\lambda) + Q_{n-1}(\lambda)R_n(\lambda), \quad \text{...............................(26)}$$

where $R_n(\lambda)$ is given by (17), and define $P_n(\lambda)$, $Q_n(\lambda)$ by the relations

$$4\Delta(\lambda - \lambda_{n-1})P_n(\lambda) = X_n(\lambda), \quad \text{.............................(27)}$$

$$4\Delta(\lambda - \lambda_{n-1})Q_n(\lambda) = Y_n(\lambda), \quad \text{.............................(28)}$$

and prove that $P_n(\lambda)$ and $Q_n(\lambda)$ satisfy the required conditions. We have to prove (*a*) that $P_n(\lambda)$ and $Q_n(\lambda)$ are in fact polynomials, *i.e.* that $\lambda - \lambda_{n-1}$ divides $X_n(\lambda)$ and $Y_n(\lambda)$; (*b*) that $P_n(\lambda)$ and $Q_n(\lambda)$ satisfy (12); (*c*) that $P_n(\lambda)$ and $Q_n(\lambda)$ have no common factor; (*d*) that the degrees of $P_n(\lambda)$ and $Q_n(\lambda)$ satisfy the prescribed inequalities. These hold for $n = 4$, by what has been said. We therefore assume them for suffix $n - 1$ and prove them for n. Our theorem then follows by induction.

(*a*) Since (25)–(28) hold with n replaced by $n - 1$, we see that

$$\begin{aligned}
4\Delta(\lambda - \lambda_{n-2})X_n(\lambda) &= X_{n-1}(\lambda)R_n(\lambda) + 4\Delta\Delta(\lambda)Y_{n-1}(\lambda) \\
&= AP_{n-2}(\lambda) + 4\Delta\Delta(\lambda)BQ_{n-2}(\lambda), \\
4\Delta(\lambda - \lambda_{n-2})Y_n(\lambda) &= X_{n-1}(\lambda) + Y_{n-1}(\lambda)R_n(\lambda) \\
&= BP_{n-2}(\lambda) + AQ_{n-2}(\lambda),
\end{aligned}$$

where

$$A = R_{n-1}(\lambda)R_n(\lambda) + 4\Delta\Delta(\lambda), \quad B = R_{n-1}(\lambda) + R_n(\lambda).$$

Now, from (17) and (10),

$$\begin{aligned} A &= [(\Theta - \Delta'\lambda_{n-2}\lambda_{n-1})\lambda + 2\Delta][(\Theta - \Delta'\lambda_{n-1}\lambda_n)\lambda + 2\Delta] + 4\Delta(\Delta'\lambda^3 + \Theta'\lambda^2 + \Theta\lambda + \Delta) \\ &= 4\Delta\Delta'\lambda^3 + [4\Delta\Theta' + \Theta^2 - \Theta\Delta'\lambda_{n-1}(\lambda_n + \lambda_{n-2}) + \Delta'^2\lambda_n\lambda_{n-2}\lambda^2_{n-1}]\lambda^2 \\ &\qquad + [8\Delta\Theta - 2\Delta\Delta'\lambda_{n-1}(\lambda_n + \lambda_{n-2})]\lambda + 8\Delta^2 \\ &= 4\Delta\Delta'\lambda^3 + \left[4\Delta\Theta' + \Theta^2 - \Theta\left(2\Theta + \frac{4\Delta}{\lambda_{n-1}}\right) + (\Theta^2 - 4\Delta\Theta' - 4\Delta\Delta'\lambda_{n-1})\right]\lambda^2 \\ &\qquad + \left[8\Delta\Theta - 2\Delta\left(2\Theta + \frac{4\Delta}{\lambda_{n-1}}\right)\right]\lambda + 8\Delta^2 \\ &= 4\Delta\Delta'\lambda^3 + \left[-4\Delta\Delta'\lambda_{n-1} - \frac{4\Delta\Theta}{\lambda_{n-1}}\right]\lambda^2 + \left[4\Delta\Theta - \frac{8\Delta^2}{\lambda_{n-1}}\right]\lambda + 8\Delta^2 \\ &= \frac{4\Delta}{\lambda_{n-1}}(\Delta'\lambda_{n-1}\lambda^2 - \Theta\lambda - 2\Delta)(\lambda - \lambda_{n-1}), \end{aligned}$$

and

$$\begin{aligned} B &= [(\Theta - \Delta'\lambda_{n-2}\lambda_{n-1})\lambda + 2\Delta] + [(\Theta - \Delta'\lambda_{n-1}\lambda_n)\lambda + 2\Delta] \\ &= [2\Theta - \Delta'\lambda_{n-1}(\lambda_n + \lambda_{n-2})]\lambda + 4\Delta = \left[2\Theta - \left(2\Theta + \frac{4\Delta}{\lambda_{n-1}}\right)\right]\lambda + 4\Delta \\ &= -\frac{4\Delta}{\lambda_{n-1}}(\lambda - \lambda_{n-1}). \end{aligned}$$

Hence $\lambda - \lambda_{n-1}$ divides A and B. It therefore divides $(\lambda - \lambda_{n-2})X_n(\lambda)$ and $(\lambda - \lambda_{n-2})Y_n(\lambda)$. Hence since, if S and S' are arbitrary, $\lambda_{n-1} \neq \lambda_{n-2}$, $\lambda - \lambda_{n-1}$ divides $X_n(\lambda)$ and $Y_n(\lambda)$, which proves (*a*).

(*b*) By the inductive hypothesis,

$$[P_{n-1}(\lambda)]^2 - 4\Delta\Delta(\lambda)[Q_{n-1}(\lambda)]^2 = d_{n-1}\lambda^{n-2}(\lambda - \lambda_{n-1}),$$

and, since (8) holds with $q = \lambda_{n-1}$, $r = \lambda_n$,

$$[R_n(\lambda)]^2 - 4\Delta\Delta(\lambda) = -4\Delta\Delta'\lambda(\lambda - \lambda_{n-1})(\lambda - \lambda_n).$$

Multiplying these identities together and using (14) with $P = P_{n-1}(\lambda)$, $Q = Q_{n-1}(\lambda)$, $R = R_{n-1}(\lambda)$, $S = 1$, we obtain, by (25) and (26),

$$[X_n(\lambda)]^2 - 4\Delta\Delta(\lambda)[Y_n(\lambda)]^2 = -4\Delta\Delta' d_{n-1}\lambda^{n-1}(\lambda - \lambda_{n-1})^2(\lambda - \lambda_n).$$

Hence, from (27) and (28),

$$[P_n(\lambda)]^2 - 4\Delta\Delta(\lambda)[Q_n(\lambda)]^2 = d_n\lambda^{n-1}(\lambda - \lambda_n),$$

where $d_n = -\Delta' d_{n-1}/4\Delta$. This proves (*b*).

(*c*) Since $P_n(\lambda)$ and $Q_n(\lambda)$ satisfy (12), their only possible common factor is a power of λ. It is therefore sufficient to show that $P_n(0)$ and $Q_n(0)$ are not zero. Now, since $R_n(0) = 2\Delta$, we have from (25)–(28)

$$\begin{aligned} -2\lambda_{n-1}P_n(0) &= P_{n-1}(0) + 2\Delta Q_{n-1}(0), \\ -4\Delta\lambda_{n-1}Q_n(0) &= P_{n-1}(0) + 2\Delta Q_{n-1}(0); \end{aligned}$$

and, from (16) and (24),

$$P_3(0) = 2\Delta Q_3(0) = 2\Delta, \quad P_4(0) = 2\Delta Q_4(0) = -2\Delta/\lambda_3.$$

Whence we find without difficulty

$$P_n(0) = 2\Delta Q_n(0) = (-1)^{n-1} \cdot 2\Delta/\lambda_3\lambda_4 \ldots \lambda_{n-1} \neq 0.$$

This proves (*c*).

The verification of (d) presents no difficulty and is left to the reader. The proof of the theorem is now complete.

5. The theorem just proved is the crucial step in the determination of λ_n, and the remainder of the work is comparatively simple. We observe, first, that by absorbing suitable constant factors, we can * write (12) in the form

$$[A_n(\lambda)]^2 - \Delta(\lambda)[B_n(\lambda)]^2 \equiv \lambda^{n-1}(\lambda - \lambda_n), \quad\ldots\ldots(29)$$

where $A_n(\lambda)$ and $B_n(\lambda)$ are polynomials with non-vanishing constant terms whose respective degrees do not exceed k, $k-1$ if $n = 2k+1$ or $k+1$, $k-1$ if $n = 2k+2$. If we define the power series $C(\lambda)$ by (2) we can write (29) in the form

$$[A_n(\lambda) + B_n(\lambda)C(\lambda)][A_n(\lambda) - B_n(\lambda)C_n(\lambda)] \equiv \lambda^{n-1}(\lambda - \lambda_n), \quad\ldots\ldots(30)$$

and, since the constant terms in $A_n(\lambda)$ and $B_n(\lambda)$ are not zero, one of the factors on the left of (30) must be a power series with a non-zero constant term and the other must be a power series whose terms all have the exponent of λ greater than or equal to $n-1$. Since $B_n(\lambda)$ is not determined as to sign by (29) we may thus suppose that

$$A_n(\lambda) + B_n(\lambda)C(\lambda) = \lambda^{n-1}(p_0 + p_1\lambda + \ldots), \quad\ldots\ldots(31)$$

$$A_n(\lambda) - B_n(\lambda)C(\lambda) = q_0 + q_1\lambda + \ldots, \quad\ldots\ldots(32)$$

where $q_0 \neq 0$ and where, since the product of the two factors is $\lambda^{n-1}(\lambda - \lambda_n)$, we have

$$p_0 q_0 = -\lambda_n. \quad\ldots\ldots(33)$$

We shall now see that λ_n can be determined by equating coefficients. We distinguish two cases, according to the parity of n.

(i) Let $n = 2k+1$, and write

$$A_n(\lambda) = \sum_{r=0}^{k} a_r\lambda^r, \quad B_n(\lambda) = \sum_{r=0}^{k-1} b_r\lambda^r, \quad C(\lambda) = \sum_{0}^{\infty} c_r\lambda^r.$$

We substitute these values in (31) and (32). Equating constant terms we see that

$$a_0 + b_0c_0 = 0, \quad a_0 - b_0c_0 = q_0,$$

and so

$$q_0 = -2b_0c_0. \quad\ldots\ldots(34)$$

Equating coefficients of $\lambda^{k+1}, \lambda^{k+2}, \ldots, \lambda^{2k}$ in (31), we have

$$b_{k-1}c_2 + b_{k-2}c_3 + \ldots + b_0c_{k+1} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$b_{k-1}c_k + b_{k-2}c_{k+1} + \ldots + b_0c_{2k-1} = 0,$$

$$b_{k-1}c_{k+1} + b_{k-2}c_{k+2} + \ldots + b_0c_{2k} = p_0.$$

From this system of equations it follows that, with the notation of (3),

$$b_{k-1} = (-1)^k \frac{u_{2k-1}}{u_{2k}} p_0, \quad b_0 = \frac{u_{2k-2}}{u_{2k}} p_0, \quad\ldots\ldots(35)$$

and by equating coefficients of λ^n in (29) we obtain

$$-\Delta' b_{k-1}^2 = 1. \quad\ldots\ldots(36)$$

From (33), (34), (35), (36) it follows, since $c_0 = \Delta^{\frac{1}{2}}$, that

$$\lambda_{2k+1} = -\frac{2\Delta^{\frac{1}{2}}}{\Delta'} \frac{u_{2k-2}u_{2k}}{u_{2k-1}^2}. \quad\ldots\ldots(37)$$

* Since $d_n \neq 0$.

(ii) Let $n = 2k+2$, and write

$$A_n(\lambda) = \sum_{r=0}^{k+1} a_r\lambda^r, \quad B_n(\lambda) = \sum_{r=0}^{k-1} b_r\lambda^r, \quad C(\lambda) = \sum_{0}^{\infty} c_r\lambda^r.$$

We substitute these values in (31) and (32). Equating constant terms we obtain (34) as before. Equating coefficients of $\lambda^{k+1}, \lambda^{k+2}, \ldots, \lambda^{2k+1}$ in (31), we find

$$\begin{aligned} a_{k+1} + b_{k-1}c_2 + \ldots + b_0c_{k+1} &= 0, \\ b_{k-1}c_3 + \ldots + b_0c_{k+2} &= 0, \\ &\ldots\ldots\ldots\ldots \\ b_{k-1}c_{k+1} + \ldots + b_0c_{2k} &= 0, \\ b_{k-1}c_{k+2} + \ldots + b_0c_{2k+1} &= p_0. \end{aligned}$$

From these equations we find

$$a_{k+1} = (-1)^{k+1}\frac{u_{2k}}{u_{2k+1}}p_0, \quad b_0 = \frac{u_{2k-1}}{u_{2k+1}}p_0, \quad \ldots\ldots\ldots\ldots (35)'$$

and equating coefficients of λ^n in (29) we obtain

$$a_{k+1}^2 = 1. \quad \ldots\ldots\ldots\ldots (36)'$$

From (33), (34), (35)′, (36)′ now follows

$$\lambda_{2k+2} = 2\Delta^{\frac{1}{2}}\frac{u_{2k-1}u_{2k+1}}{u_{2k}^2}. \quad \ldots\ldots\ldots\ldots (37)'$$

Thus λ_n is given by (37) or (37)′ according as n is odd or even, as was stated at (11).

In particular, the condition that an n-gon exists inscribed in S' and circumscribed to S is given by $\lambda_n = 0$, $\lambda_{n-2} \neq 0$, and this leads to the condition $u_{n-1} = 0$, as stated by Cayley. J. A. T.

THE "GAZETTE": AN ALTERATION.

As a measure of economy in distribution, the Council of the Association has decided that the annual volumes of the *Gazette* shall, for the immediate future, be issued in four parts instead of five. Since no diminution in the size of the volume is intended, this alteration must be regarded as a minor re-arrangement which will not affect the amount of material contained in the *Gazette*. It is hoped that the alteration may be temporary only, to be reviewed as soon as the financial circumstances of the Association permit.

GLEANINGS FAR AND NEAR.

1582. A mathematician once said that algebra was the science for lazy people—one does not work out x, but operates with it as if one knew it. In our case, x stands for the anonymous masses, the people. Politics means operating with this x without worrying about its actual nature. Making history is to recognise x for what it stands for in the equation.—A. Koestler, *Darkness at Noon*. [Per Mr. E. S. Davis.]

1583. Dave Albritton, an American negro, who also jumped 6 ft. 9¾ in. in 1936, introduced something entirely new with the Straddle jump. This made the jumper's navel the nearest point to the bar and therefore his centre of gravity.—F. A. M. Webster, "How high can a man jump?" *Country Life*, July 11, 1947. [Per Mr. J. B. Marriott.]

MODULATION AND HETERODYNE ACTION.

By A. Ladle and J. W. Whitehead.

A Simple Mathematical Analysis of Modulation and Heterodyne Action between Two Sinusoidal Oscillations, indicating Resemblances between the Resultant Waves of the Two Types.

Introduction.

The contents of this paper will, it is hoped, be found to be of value to teachers of physics and radio principles, to their pupils, and to those engineers who prefer a full understanding of phenomena to a superficial knowledge.

It is an unfortunate fact that the mechanism of beats and of modulation is sometimes ill-understood, and this is perhaps because ignorance of the mathematical principles involved prevents a true understanding of the subject. A particular case in point is the frequent difficulty of the student to distinguish between the resultant of two beating sinusoidal oscillations of different amplitudes but approximately equal frequencies, and a sinusoidal oscillation of low frequency modulating a high frequency vibration of larger amplitude —merely because the diagrammatic likenesses are great.

In the analysis which follows, an attempt is made to point out the differences, and at the same time to reveal the resemblance between the beating of two oscillations of roughly equal frequency and a modulated wave.

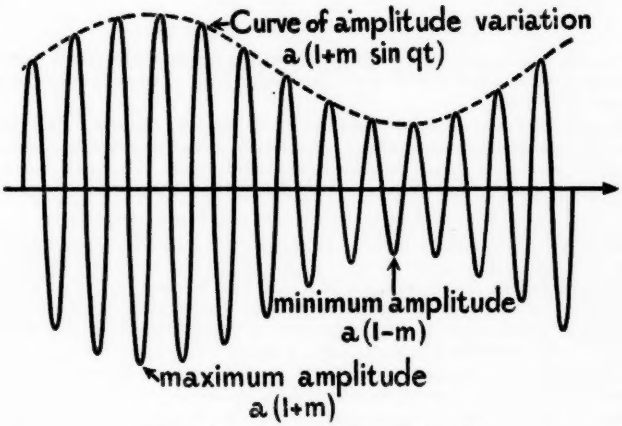


Fig. 1 (*a*). A modulated wave ($m < 1$).

(*a*) *Simple modulation.*

Suppose a sinusoidal oscillation of angular frequency p to be modulated by another of angular frequency q. By the modulation of one with the other we mean that to the oscillation $a \sin pt$ is added another, $ma \sin pt \,.\, \sin qt$ (m being a quantity called the " depth of modulation "), and so we obtain the resultant oscillation :

$$a(1 + m \sin qt) \sin pt = a \sin pt + \tfrac{1}{2}am \{\cos (p-q)t - \cos (p+q)t\}, \quad (p > q),$$

i.e. the modulation of one sinusoidal oscillation with another results in an oscillation consisting of three components with equally spaced frequencies :

$p/2\pi$, $(p-q)/2\pi$, $(p+q)/2\pi$, the first being the " carrier wave ", and the second and third the " lower " and " upper " sidebands respectively.

A diagrammatic representation of this waveform is shown in Fig. 1*a*. In general, m is less than unity, but in the case of $m=1$ (corresponding to 100% modulation) the minimum amplitude is zero. If m exceeds unity, the diagram assumes the shape of Fig. 1*b*, and two zero values appear. At these zero

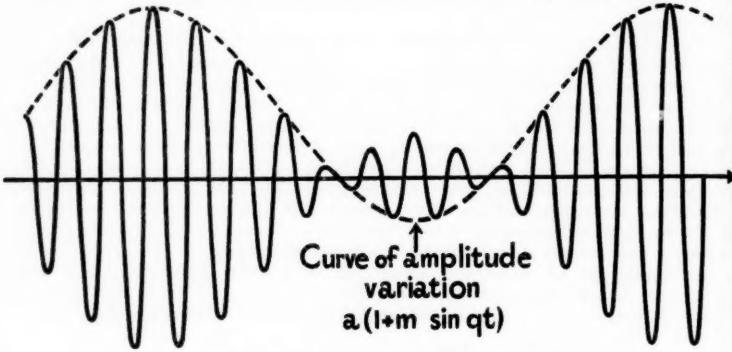


FIG. 1 (*b*). A modulated wave ($m>1$).

values a phase change occurs causing part of the curve of amplitude variation which was on the upper side of the mean line in Fig. 1*a* to switch to the lower side, and *vice versa*.

(*b*) *The beating of two sinusoidal oscillations of equal amplitudes.* (Fig. 2).

Now let two oscillations of equal amplitudes and of angular frequencies p_1 and q_1, $((p_1+q_1) \gg (p_1-q_1))$ be arranged to heterodyne. The resultant at any instant (neglecting " phase " terms) is given by E, where

$$E = a \sin p_1 t + a \sin q_1 t = 2a \sin \tfrac{1}{2}(p_1+q_1)t \,.\, \cos \tfrac{1}{2}(p_1-q_1)t,$$

[Hereafter we will generally write α for $\frac{1}{2}(p_1+q_1)t$ and β for $\frac{1}{2}(p_1-q_1)t$.]

This is a case of beating at $(p_1-q_1)/2\pi$ hertz, and we have, in effect, an amplitude variation $2a \cos \frac{1}{2}(p_1-q_1)t$ which has turning points at intervals of $2\pi/(p_1-q_1)$, that is, a maximum irrespective of sign, when $\cos\beta=1$ (Fig. 2). (It will be noted that, in this particular case of equal amplitudes, the resultant frequency is the arithmetic mean of the component frequencies.)

(c) *The beating of two sinusoidal oscillations of unequal amplitudes.*

Heterodyne actions encountered in the radio technique are rarely the consequence of oscillations of equal amplitudes. More often than not, they are widely dis-similar. It is therefore opportune to consider beating between oscillations of unequal amplitudes.

Let the heterodyning waveforms be $a \sin p_1 t$ and $b \sin q_1 t$. The resultant E due to their superposition will be :

$$\begin{aligned} E &= a \sin p_1 t + b \sin q_1 t \quad \text{.......(i)} \\ &= a \sin(\alpha+\beta) + b \sin(\alpha-\beta) \\ &= a \sin\alpha \,.\, \cos\beta + a \cos\alpha \,.\, \sin\beta + b \sin\alpha \cos\beta - b \cos\alpha \,.\, \sin\beta \\ &= (a+b)\cos\beta \,.\, \sin\alpha + (a-b)\sin\beta \,.\, \cos\alpha \quad \text{.......(ii)} \\ &= A \sin\alpha + B\cos\alpha \end{aligned}$$

where $A=(a+b)\cos\beta$ and $B=(a-b)\sin\beta$.

Thus $E=\sqrt{(A^2+B^2)}\,.\sin(\alpha+\phi)$ where $\phi=\tan^{-1}B/A$.

B and A are functions of time, and so it can be said that the resultant of two sine waves beating is another wave of variable amplitude and phase.

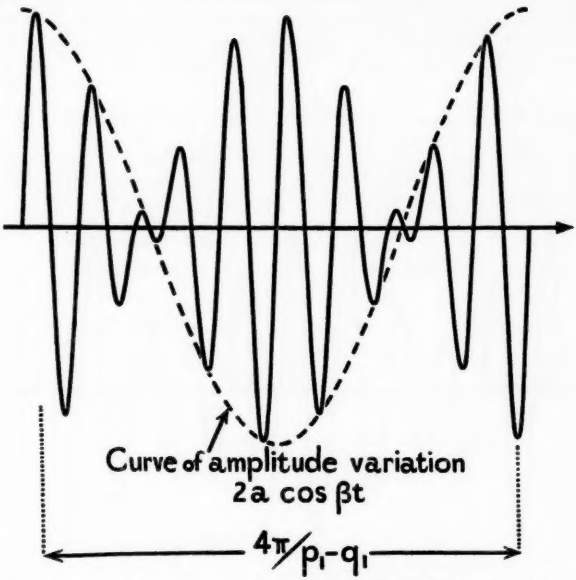


FIG. 2. Beating of two oscillations of equal amplitudes $(p_1+q_1 \gg p_1-q_1)$.

(*d*) *To find an expression for the resultant amplitude variation of two sinusoidal oscillations of different amplitudes.*

It is possible, without much difficulty, to obtain an expression for the amplitude variation in this particular case, and it is of interest because such conditions frequently obtain in the radio technique, *e.g.* in the heterodyne reception of C.W. signals, in tone generators working on the beat principle, and sometimes in the mixer stage of a superheterodyne receiver.

Using the notations of sections (*b*) and (*c*), E is a maximum when $dE/dt=0$, *i.e.* when $$ap_1\cos p_1t+bq_1\cos q_1t=0,$$

$$\tfrac{1}{2}(ap_1+bq_1)(\cos p_1t+\cos q_1t)+\tfrac{1}{2}(ap_1-bq_1)(\cos p_1t-\cos q_1t)=0,$$

$$(ap_1+bq_1)\cos\alpha\,.\cos\beta-(ap_1-bq_1)\sin\alpha\,.\sin\beta=0,$$

whence $\tan\alpha=\lambda\cot\beta$, where $\lambda=(ap_1+bq_1)/(ap_1-bq_1)$. ....(iii)

Any curve through the maxima of (i) must satisfy (ii) and (iii). Eliminating α from (ii) and (iii), we obtain :

$$F=(a+b)\frac{\lambda\cot\beta}{\sqrt{(1+\lambda^2\cot^2\beta)}}\cos\beta+(a-b)\,.\frac{1}{\sqrt{(1+\lambda^2\cot^2\beta)}}\sin\beta, \quad \text{.....(iv)}$$

[the sine and cosine being derived from the tangent, the same sign is taken for the root in each case].

If $a \gg b$ and p_1 is of the same order as q_1, we may make the approximation

$$\lambda = \frac{ap_1 + bq_1}{ap_1 - bq_1} = 1,$$

and then

$$F = (a+b)\frac{\cot\beta}{\sqrt{1+\cot^2\beta}} \cdot \cos\beta + (a-b)\frac{1}{\sqrt{1+\cot^2\beta}} \cdot \sin\beta,$$

whence

$$F = (a+b)\cos^2\beta + (a-b)\sin^2\beta = a + b(\cos^2\beta - \sin^2\beta) = a + b\cos(p_1 - q_1)t, \quad \text{...(v)}$$

which represents exactly the amplitude variation of a wave modulated to a depth b/a by a cosine wave $b\cos(p_1 - q_1)t$ (Fig. 3). In other words, the amplitude variation in these particular circumstances is practically "pure".

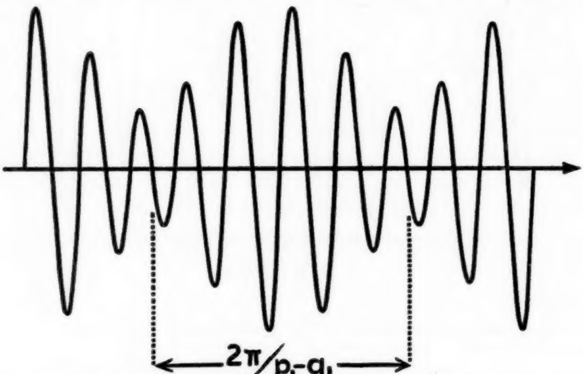


FIG. 3. Heterodyning with very dissimilar amplitudes but roughly equal frequencies. [Amplitude variation $= a + b\cos(p_1 - q_1)t$.]

From (iv) we may obtain the equation in the form

$$F = \frac{(a+b)\lambda\cos^2\beta + (a-b)\sin^2\beta}{\sqrt{(\sin^2\beta + \lambda^2\cos^2\beta)}}.$$

If no limitations are placed on a and b, but it is assumed that

$$(p_1 - q_1) \ll (p_1 + q_1),$$

we may make the assumption

$$\lambda = \frac{ap_1 + bq_1}{ap_1 - bq_1} = \frac{a+b}{a-b},$$

whence

$$F = \sqrt{\{(a+b)^2\cos^2\beta + (a-b)^2\sin^2\beta\}} = \sqrt{\{a^2 + b^2 + 2ab\cos(p_1 - q_1)t\}}. \quad \text{.....(vi)}$$

This equation representing the "envelope" of the waveform when the two frequencies are very nearly equal might have been obtained from the equation given in (c) above, namely,

$$E = \sqrt{(A^2 + B^2)} \cdot \sin(\alpha + \phi).$$

Usually, $\sqrt{(A^2+B^2)}$ is taken as the " envelope ", but in view of the fact that ϕ is not constant (being a function of time), to assume it to be constant would appear unjustifiable. (It is realised that true mathematicians will detect a weakness here.)

By considering the conditions specified in particular cases, it is possible to simplify equation (vi) to forms identical with those already arrived at.

For example, when the heterodyning oscillations are of equal amplitudes (*i.e.* $a=b$), then

$$F=\sqrt{\{a^2+a^2+2aa\cos(p_1-q_1)t\}}=\sqrt{2}\,.\,a(1+\cos^2\beta-\sin^2\beta)^{\frac{1}{2}}=\pm 2a\cos\beta,$$

i.e. the " envelope " is a half-cosine curve of maximum amplitude $2a$ and half the frequency difference. (Refer back to section (*b*) and Fig. 2.)

If b is very small compared with a (*i.e.* a weak oscillation is heterodyned with a strong one), b^2 is much smaller than a^2 and equation (vi) becomes :

$$\begin{aligned}F&=\{a^2+2ab\cos(p_1-q_1)t\}^{\frac{1}{2}}\\&=(a^2)^{\frac{1}{2}}+\tfrac{1}{2}(a^2)^{-\frac{1}{2}}\,.\,2ab\cos(p_1-q_1)t+\text{negligible terms}\\&=a+b\cos(p_1-q_1)t\,;\end{aligned}$$

or, the " envelope " approximates to a cosine curve when one oscillation is small compared with the other. (Refer back to equation (v).)

If no limits are placed on the frequencies of the beating oscillations, an expression for the amplitude variation may be obtained as follows. Equation (vi) may be written in the form :

$$F=\theta(1+\delta\cos\gamma)^{\frac{1}{2}},\quad\text{.............................(vii)}$$

where $\theta=\sqrt{(a^2+b^2)}$, $\delta=2ab/(a^2+b^2)$, $\gamma=(p_1-q_1)t$.

Expanding equation (vii),

$$F=\theta\left\{1+\frac{\delta}{2}\cos\gamma-\frac{\delta^2}{8}\cos^2\gamma+\frac{\delta^3}{16}\cos^3\gamma\ldots\right.$$
$$\left.\frac{(-1)^{r-1}\,.\,1\,.\,3\,.\,5\ldots(2r-3)}{2^r\,.\,r!}\,.\,\delta^r\,.\cos^r\gamma\ldots\right\},$$

and assuming the expressions for the power of a cosine (see Hobson, *Plane Trigonometry*, pp. 53–54) :

$$F=\theta\left\{1+\frac{\delta}{2}\cos\gamma-\frac{\delta^2}{8}\left[\tfrac{1}{2}(1+\cos 2\gamma)\right]+\frac{\delta^3}{16}\left[\tfrac{1}{4}(3\cos\gamma+\cos 3\gamma)\right]\right.$$
$$\left.+\frac{(-1)^{r-1}\,.\,1\,.\,3\,.\,5\ldots(2r-3)\,\delta^r}{2^r\,.\,r!}\left[\frac{1}{2^{r-1}}(\cos r\gamma+r\cos\overline{r-2}\,.\,\gamma+\ldots)\right]+\ldots\right\},$$

$$F=\theta\left\{\left(1-\frac{\delta^2}{16}-\frac{15}{1024}\delta^4-\frac{105}{16{,}384}\delta^5\ldots\right)+\left(\frac{\delta}{2}+\frac{3}{64}\delta^3+\frac{35}{2048}\delta^5+\ldots\right)\cos\gamma\right.$$
$$\left.-\left(\frac{1}{16}\delta^2+\frac{5}{256}\delta^4+\frac{315}{32{,}768}\delta^6+\ldots\right)\cos 2\gamma+\ldots\right\},$$

$$\begin{aligned}F=\sqrt{a^2+b^2}\{&(1-0{\cdot}0625\delta^2-0{\cdot}01466\delta^4-0{\cdot}0064\delta^6\ldots)\\&+(0{\cdot}5\delta+0{\cdot}0469\delta^3+0{\cdot}0171\delta^5+\ldots)\cos(p_1-q_1)t\\&-(0{\cdot}0625\delta^2+0{\cdot}0196\delta^4+0{\cdot}0096\delta^6+\ldots)\cos 2(p_1-q_1)t\\&+(0{\cdot}0156\delta^3+0{\cdot}0082\delta^5+\ldots)\cos 3(p_1-q_1)t+\text{etc.}\end{aligned}$$

This series indicates completely the waveform of the resultant amplitude variation of two beating oscillations of roughly equal frequencies. From what has gone before, it is evident that it must also represent the modulation frequency of the equivalent modulated wave.

In short, two beating sinusoidal oscillations of comparable frequency may be regarded as being the equivalent of another wave of about the mean frequency of the beating waves modulated by an oscillation which is not sinusoidal unless the component amplitudes are very dissimilar.

It is worthy of note that, owing to the fact that ϕ derived from equation (ii) is a function of time, the equivalent modulated wave is, to some extent, frequency modulated at the beat frequency; in fact, the instantaneous angular frequency ω is given by :

$$\omega=\frac{d}{dt}\left(\frac{p_1+q_1}{2}t+\phi\right)=\frac{a^2p_1+b^2q_1+ab(p_1+q_1)\cos(p_1-q_1)t}{a^2+b^2+2ab\cos(p_1-q_1)t}$$

$$\simeq p_1\left[1-\frac{b^2}{a^2}\left(\frac{p_1-q_1}{p_1}\right)-\frac{b}{a}\left(\frac{p_1-q_1}{p_1}\right)\cos(p_1-q_1)t\right]$$

which, if $b/a \ll 1$, and $\frac{p_1-q_1}{p_1} \ll 1$, gives

$$\omega \simeq p_1\left[1-\frac{b}{a}\left(\frac{p_1-q_1}{p_1}\right)\cos(p_1-q_1)t\right],$$

i.e. ω varies between the limits

$$p_1\left[\quad -\frac{b}{a}\left(\frac{p_1-q_1}{p_1}\right)\right] \quad \text{and} \quad p_1\left[1+\frac{b}{a}\left(\frac{p_1-q_1}{p_1}\right)\right].$$

In other words, the mean frequency of the equivalent carrier tends to p_1 rather than to $(p_1+q_1)/2$. In fact, $\omega=(p_1+q_1)/2$ only when $a=b$.

ACKNOWLEDGMENT.

The authors are indebted to Mr. F. Caine of Leeds Central High School for the preparation of the diagrams for this paper.

A. L.
J. W. W.

1584. A dividend . . . of approximately $5\frac{9}{32}$ or, more precisely, 5·282158 per cent.—*The Times*, Feb. 20, 1948.

1585. In the vicinity of Madras there must be between six and seven hundred spotted owlets to the square mile, so that, if their voices were audible three miles away, and all spoke at once, we should spend our nights listening to a chorus of about two thousand spotted owlets.—Douglas Dewar, *Bombay Ducks, or Birds and Beasts found in a Naturalist's El Dorado.* [Per Mr. C. B. Gordon.]

1586. Under the mere instinctive routine which I shared with the bee or the coral insect, there was a real core of purpose, of my own purpose and not anybody else's, partly inherited but in great part maintained, if not actually created, by a sheer effort of will. That was what St. Thomas Aquinas might have called the true spark of me, my ".scintilla" ; or the " godly purpose " of Juliana of Norwich. That scintilla utilized my natural sensitiveness to local associations : the sudden flash of " Here is something you have come across before " : with a restless urge to catch and recall this half-forgotten image. Such are the clues which lead us on from the merely arithmetical to the geometrical stage of thought ; the little floating germ takes stature and shape from this logical association with its new fellow.—G. G. Coulton, *Four Score Years* (C.U.P., 1944), p. 303. [Per Mr. J. Buchanan.]

SOME EXERCISES IN LAPLACE TRANSFORM INTEGRALS.

By John Williams.

If $\phi(t)$ is a given function of the real variable t then the integral (in which p is constant)

$$\mathbf{L}\{\phi(t)\} = \int_0^\infty e^{-pt}\phi(t)\,dt = \Phi(p) \quad \ldots\ldots(1)$$

if convergent, is defined to be the Laplace transform of $\phi(t)$. The use of the integral in determining the solutions of certain types of differential equations is now well known; see, for example, Carslaw and Jaeger, *Operational methods in applied mathematics* (Oxford University Press).

Let us consider the integrals

$$\int_0^\infty e^{-pt} \begin{matrix}\sin\\ \cos\end{matrix}\, \alpha t \,.\, \phi(t)\,dt, \quad \ldots\ldots(2)$$

where both p and α are real (and positive). Considering p as fixed these integrals define functions of α, continuous in a region $0 \leqslant \alpha \leqslant \lambda$ if

$$\left.\begin{array}{l}\text{(i) } \phi(t) \text{ is a continuous function of } t \text{ for all } t \geqslant 0,\\ \text{(ii) the integrals are uniformly convergent for } 0 \leqslant \alpha \leqslant \lambda.\end{array}\right\} \quad \ldots\ldots(3)$$

These conditions are obviously sufficient but not in general necessary. When the integrals in (2) are convergent and define continuous functions of α for $0 \leqslant \alpha \leqslant \lambda$, then

$$\begin{aligned}\mathbf{L}\{\cos\alpha t\,.\,\phi(t)\} &= \int_0^\infty e^{-pt}\cos\alpha t\,.\,\phi(t)\,dt\\ &= \mathbf{R}\int_0^\infty e^{-(p-i\alpha)t}\phi(t)\,dt\\ &= \mathbf{R}\Phi(p-i\alpha),\\ \mathbf{L}\{\sin\alpha t\,.\,\phi(t)\} &= \mathbf{I}\Phi(p-i\alpha).\end{aligned} \quad \ldots\ldots(4)$$

Sufficient conditions for the uniform convergence of the integrals in (2) can be stated by an appeal to either the de la Vallée Poussin or the Chartier test, for which we refer the reader to the standard treatises on analysis. In fact, from the former test we have the condition of sufficiency that the integrals in (2) are uniformly convergent in any region of α when

$$\int_0^\infty e^{-pt}\,|\,\phi(t)\,|\,dt \quad \ldots\ldots(5)$$

is convergent. Again, from the Chartier test, since

$$\left|\int_0^t e^{-kt}\begin{matrix}\cos\\ \sin\end{matrix}\,\alpha t\,.\,dt\right|$$

is uniformly bounded as $t \to \infty$ when $k > 0$ for every region of α, the condition that either integral in (2) should be uniformly convergent in any region of α reduces to the condition that

$$e^{-(p-k)t}\phi(t) \to 0 \text{ steadily as } t \to \infty. \quad \ldots\ldots(6)$$

Obviously finer tests than these can be designed to meet the needs of a special problem. However, the tests stated are sufficient to meet the needs of the ensuing examples.

A further generalisation of result (4) can be made by obtaining the expressions for

$$\mathbf{L}\{\cos^n\alpha t\,.\,\phi(t)\}, \quad \mathbf{L}\{\sin^n\alpha t\,.\,\phi(t)\},$$

where n is a positive integer. In fact, using the expansions for $\cos^n \alpha t$, $\sin^n \alpha t$ in terms of the sines and cosines of their multiple angles we have, subject to certain conditions of validity, that

$$2^{2k}\mathbf{L}\{\cos^{2k}\alpha t\,.\,\phi(t)\} = {}^{2k}C_k\Phi(p) + 2\mathbf{R}\sum_{r=0}^{k-1} {}^{2k}C_r\Phi[p-(k-r)\,2i\alpha], \qquad \text{(7a)}$$

$$2^{2k}\mathbf{L}\{\sin^{2k}\alpha t\,.\,\phi(t)\} = {}^{2k}C_k\Phi(p) + 2\mathbf{R}\sum_{r=0}^{k-1} (-)^{k-r}\,{}^{2k}C_r\Phi[p-(k-r)\,2i\alpha], \qquad \text{(7b)}$$

where k is an integer greater than zero, and

$$2^{2k}\mathbf{L}\{\cos^{2k+1}\alpha t\,.\,\phi(t)\} = \mathbf{R}\sum_{r=0}^{k} {}^{2k+1}C_r\Phi[p-(2k+1-2r)\,i\alpha], \qquad \text{(7c)}$$

$$2^{2k}\mathbf{L}\{\sin^{2k+1}\alpha t\,.\,\phi(t)\} = \mathbf{I}\sum_{r=0}^{k} (-)^{k-r}\,{}^{2k+1}C_r\Phi[p-(2k+1-2r)\,i\alpha], \qquad \text{(7d)}$$

where k is a positive integer or zero.

Examples.

(i) When $\phi(t)=t^n$, where n is a positive integer or zero, then $\Phi(p)=n!\,p^{-n-1}$ (*vide* Carslaw and Jaeger, *loc. cit.*). In this case $|\,\phi(t)\,|=\phi(t)$ for all $t\geqslant 0$ so that $\mathbf{L}\{|\,\phi(t)\,|\}=n!\,p^{-n-1}$, convergent for $p>0$. Since t^n is continuous for $t\geqslant 0$, it follows that $\mathbf{L}\{\cos\alpha t\,.\,t^n\}$ and $\mathbf{L}\{\sin\alpha t\,.\,t^n\}$ are uniformly convergent for all regions of α and define continuous functions of α for all regions of α. Consequently from (4) we write

$$\mathbf{L}\{\cos\alpha t\,.\,t^n\}=n!\,\mathbf{R}(p-i\alpha)^{-n-1}, \qquad \text{(8a)}$$

$$\mathbf{L}\{\sin\alpha t\,.\,t^n\}=n!\,\mathbf{I}(p-i\alpha)^{-n-1}, \qquad \text{(8b)}$$

where n is a positive integer or zero and p is positive. Using De Moivre's theorem we have, therefore,

$$\mathbf{L}\left\{\begin{matrix}\cos\\ \sin\end{matrix}\,\alpha t\,.\,t^n\right\} = (p^2+\alpha^2)^{-(n+1)/2}\begin{matrix}\cos\\ \sin\end{matrix}\,(n+1)\,\theta, \qquad \text{(9)}$$

where $\tan\theta=\alpha/p$ and $0\leqslant\theta<\tfrac{1}{2}\pi$, since $\alpha\geqslant 0$, $p>0$.

In the particular case when $n=0$, we get the well-known results

$$\mathbf{L}\{\cos\alpha t\}=p/(p^2+\alpha^2), \quad \mathbf{L}\{\sin\alpha t\}=\alpha/(p^2+\alpha^2).$$

Using relations (7), and writing

$$\lambda_r={}^{2k}C_r\,.\,[p^2+(k-r)^2\,.\,4\alpha^2]^{-1}, \quad \mu_r={}^{2k+1}C_r\,.\,[p^2+(2k+1-2r)^2\alpha^2]^{-1},$$

we obtain

$$2^{2k}\mathbf{L}\{\cos^{2k}\alpha t\}={}^{2k}C_k\,p^{-1}+2p\sum_{r=0}^{k-1}\lambda_r, \quad (k\geqslant 1), \qquad \text{(10a)}$$

$$2^{2k}\mathbf{L}\{\sin^{2k}\alpha t\}={}^{2k}C_k\,p^{-1}+2p\sum_{r=0}^{k-1}(-)^{k-r}\lambda_r, \quad (k\geqslant 1), \qquad \text{(10b)}$$

$$2^{2k}\mathbf{L}\{\cos^{2k+1}\alpha t\}=p\sum_{r=0}^{k}\mu_r, \quad (k\geqslant 0), \qquad \text{(10c)}$$

$$2^{2k}\mathbf{L}\{\sin^{2k+1}\alpha t\}=\alpha\sum_{r=0}^{k}(-)^{k-r}(2k+1-2r)^{2k+1}\mu_r, \quad (k\geqslant 0), \qquad \text{(10d)}$$

where in each case k is an integer.

(ii) When $\phi(t)=J_n(t)$, where n is a positive integer or zero, then

$$\Phi(p)=(p^2+1)^{-\frac{1}{2}}[\sqrt{(p^2+1)}-p]^n, \qquad \text{(11)}$$

(*vide* Carslaw and Jaeger, *loc. cit.*). To examine the uniform convergence of $\mathbf{L}\{\cos \alpha t \,.\, J_n(t)\}$ we resort to the well-known integral formula for the Bessel functions of positive integral or zero order, namely,

$$\pi J_n(t) = \int_0^\pi \cos(n\theta - t\sin\theta)\,d\theta.$$

Now $$\pi \mid J_n(t) \mid \leqslant \int_0^\pi \mid \cos(n\theta - t\sin\theta) \mid d\theta,$$

and hence if t is real

$$\pi \mid J_n(t) \mid \leqslant \int_0^\pi 1 \,.\, d\theta = \pi.$$

It follows, therefore, that

$$\int_0^\pi e^{-pt} \mid J_n(t) \mid dt \leqslant \int_0^\infty e^{-pt} dt = p^{-1},$$

and is convergent when $p > 0$; thus, $\mathbf{L}\{\cos \alpha t \,.\, J_n(t)\}$ and $\mathbf{L}\{\sin \alpha t \,.\, J_n(t)\}$ are uniformly convergent for any region of α. Since $J_n(t)$ is a continuous function of t for all t, these integrals also define continuous functions of α for all regions of α. Consequently we write

$$\mathbf{L}\{\cos \alpha t \,.\, J_n(t)\} = \mathbf{R}\{[(p - i\alpha)^2 + 1]^{-\frac{1}{2}}[(\sqrt{\{(p - i\alpha)^2 + 1\}} - p + i\alpha]^n\}, \quad \text{.....(12a)}$$

$$\mathbf{L}\{\sin \alpha t \,.\, J_n(t)\} = \mathbf{I}\{[(p - i\alpha)^2 + 1]^{-\frac{1}{2}}[\sqrt{\{(p - i\alpha)^2 + 1\}} - p + i\alpha]^n\}. \quad \text{.....(12b)}$$

It can easily be shown that

$$[(p - i\alpha)^2 + 1]^{-\frac{1}{2}} = \pm[\{A + (p^2 - \alpha^2 + 1)\}^{\frac{1}{2}} + i\{A - (p^2 - \alpha^2 + 1)\}^{\frac{1}{2}}]/A\sqrt{2},$$

where $$A = +\sqrt{\{(p^2 - \alpha^2 + 1)^2 + 4p^2\alpha^2\}}.$$

Choosing the positive sign in the ambiguity of the last expression to correspond to the positive sign associated with the square root in (11), we have, using (4), and putting $n = 0$ in (12)

$$\mathbf{L}\left\{\begin{matrix}\cos \\ \sin\end{matrix}\ \alpha t \,.\, J_o(t)\right\} = \frac{1}{A\sqrt{2}}[A \pm (p^2 - \alpha^2 + 1)]^{\frac{1}{2}}, \quad (p > 0), \quad \text{..........(13)}$$

where $$A = +\sqrt{\{(p^2 - \alpha^2 + 1)^2 + 4p^2\alpha^2\}}.$$

In order to obtain expressions for the more general case exhibited by (12) we put

$$+\sqrt{[\tfrac{1}{2}\{A + (p^2 - \alpha^2 + 1)\}]} = \mu, \quad +\sqrt{[\tfrac{1}{2}\{A - (p^2 - \alpha^2 + 1)\}]} = \lambda, \quad \text{........(14)}$$

so that the complex expression on the right-hand side of (12) becomes

$$(1/A)(\mu + i\lambda)[\mu - p - i(\lambda - \alpha)]^n. \quad \text{........................(15)}$$

The reader should note that the positive signs have been associated with the square roots in accordance with expression (11). If now we introduce r and θ such that

$$r\cos\theta = \mu - p, \quad r\sin\theta = \lambda - \alpha,. \quad \text{.......................(16a)}$$

we have

$$r = +\sqrt{[(\mu - p)^2 + (\lambda - \alpha)^2]}, \quad \tan\theta = (\lambda - \alpha)/(\mu - p), \quad \text{..........(16b)}$$

(since $\mu > p$) where $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Finally, using De Moivre's theorem, and separating the real and imaginary parts of (15), we write

$$A\mathbf{L}\{\cos \alpha t \,.\, J_n(t)\} = r^n(\mu\cos n\theta + \lambda\sin n\theta) \quad \text{..................(17a)}$$

$$A\mathbf{L}\{\sin \alpha t \,.\, J_n(t)\} = r^n(\lambda\cos n\theta - \mu\sin n\theta) \quad \text{..................(17b)}$$

provided $p > 0$, where λ, μ are given by (14), r, θ by (16) and

$$A = +\sqrt{[(p^2 - \alpha^2 + 1)^2 + 4p^2\alpha^2]}.$$

Other exercises can be worked similarly, for example, $\mathbf{L}\{\sin \alpha t \sinh \beta t\}$. J. W.

T

A PROBLEM ON CIRCLES.

By E. R. Reifenberg.

In connection with a general form of the covering principle and the relative differentiation of additive functions Mr. A. S. Besicovitch has proved * that given a unit circle C with centre O, then any set of circles satisfying the conditions

1. Each circle of the set meets (or touches) C;

A. 2. Each circle of the set has radius not less than 1;

3. No circle contains O or the centre of any other circle of the set,

has less than 22 members. We now prove the complete result:

Theorem 1. *Any set of circles C_i, centres O_i and radii r_i, satisfying A, has at most* 18 *members.*

To prove this we need a number of lemmas.

Lemma 1. *If there is a set of n circles satisfying A, then there is such a set of n unit circles.*

Proof. We may, without violating A, reduce the radii of the circles, keeping the centres fixed, till they are either unit circles or touch C. If now we have k_i circles of radius r_i, $r_1 > r_2 > \dots$, replace each of the circles of radius r_1 by a circle radius r_2 with the same point of contact with C as before. We obtain a new set of circles again satisfying A. We repeat this process till we are left with unit circles only.

Lemma 2. *Given two unit circles C_1, C_2 satisfying A, if* $\min(OO_1, OO_2) \geqslant \frac{3}{2}$, *then* $\angle O_1OO_2 > 28\frac{3}{4}°$.

Proof. The minimum will occur either when $O_1O_2 = 1$, $OO_1 = OO_2 = 2$, or when $O_1O_2 = 1$, $OO_1 = 2$, $OO_2 = \frac{3}{2}$. In both cases $\cos O_1OO_2 = \frac{7}{8}$, and so $\angle O_1OO_2 > 28\frac{3}{4}°$.

Corollary. There can be 12 but not more circles C_i, satisfying A, with $OO_i \geqslant \frac{3}{2}$ $(i = 1, 2, \dots)$.

Lemma 3. *Given two unit circles C_1, C_2, satisfying A, if* $\max(OO_1, OO_2) \leqslant \frac{3}{2}$, *then* $\angle O_1OO_2 > 38\frac{3}{4}°$.

Proof. $\angle O_1OO_2$ will be least either when $O_1O_2 = 1$ and $OO_1 = OO_2 = \frac{3}{2}$ or when $O_1O_2 = 1$, $OO_1 = \frac{3}{2}$, $OO_2 = 1$. In the two cases $\cos O_1OO_2 = \frac{7}{9}$ and $\frac{3}{4}$ respectively, and so $O_1OO_2 > 38\frac{3}{4}°$.

Corollary. There can be 9 *but not more circles C_i satisfying A with*

$$OO_i \leqslant \tfrac{3}{2} \ (i = 1, 2, \dots).$$

A similar proof gives:

Given two unit circles C_1, C_2 satisfying A, if $\max(OO_1, OO_2) \leqslant 1{\cdot}2$, *then* $\angle O_1OO_2 > 49\frac{1}{7}°$.

Lemma 4. *Given three unit circles C_1, C_2, C_3, satisfying A such that the vector OO_3 lies between the vectors OO_1, OO_2, and* $\max(OO_1, OO_2) \leqslant \frac{3}{2}$, $OO_3 \geqslant \frac{3}{2}$, *then* $\angle O_1OO_2 > 41\frac{1}{3}°$.

Proof. By displacing O_3 along OO_3 without altering O_1, O_2 we may suppose that C_3 touches C, $OO_3 = 2$. Also when $\angle O_1OO_2$ is a minimum O_1O_2 will be unity. We consider three cases according as:

(1) $\max(OO_1, OO_2) < \frac{3}{2}$, $\min(OO_1, OO_2) > 1$; (2) $OO_1 = 1$; (3) $OO_2 = \frac{3}{2}$.

Case 1. Both O_1, O_2 are on C_3 when $\angle O_1OO_2$ is a minimum, for if not we could turn C_3 about O till neither O_1 nor O_2 is on it and then displace the vector O_1O_2 so as to decrease $\angle O_1OO_2$. We can now displace the chord O_1O_2

* *Proc. Camb. Phil. Soc.*, Vol. 41, part 2.

of C_2 along the circle, keeping its length fixed so as to decrease $\angle O_1OO_2$, and so the minimum of $\angle O_1OO_2$ is not attained in this case.

Case 2. $OO_1 = O_1O_2 = 1, OO_2 \leqslant \frac{3}{2}$. Hence $\cos O_1OO_2 \leqslant \frac{3}{4}$ and $\angle O_1OO_2 > 41\frac{1}{3}°$.

Case 3. $O_1O_2 = 1,\ OO_3 = 2,\ OO_2 = \frac{3}{2}$. By rotating C_3 about O we may suppose that O_2 is on C_3. Now O_1O_3 decreases as OO_1 increases, and if $O_1O_3 = 1$ then $\cos O_3O_2O = -\frac{1}{4}$ and $\angle O_1O_2O = \angle O_3O_2O - 60° = 44°\ 29'$ and $OO_1 < 1{\cdot}1$. Thus $1 \leqslant OO_1 < 1{\cdot}1$, and thus $\cos O_1OO_2 = (1\frac{1}{4} + OO_1^2)/3\ .\ OO_1 \leqslant \frac{3}{4}$.
Hence $\angle O_1OO_2 > 41\frac{1}{3}°$.

Lemma 5. *Given four unit circles C_1, C_2, C_3, C_4, satisfying A, and such that the vectors OO_1, OO_2 lie between the vectors OO_3, OO_4 and* min $(OO_1, OO_2) \geqslant \frac{3}{2}$, *then*

(1) *if* max $(OO_3, OO_4) \leqslant \frac{3}{2}$, *we have* $\angle O_3OO_4 > 52\frac{1}{7}°$;

(2) *if* max $(OO_3, OO_4) \leqslant 1{\cdot}2$, *we have* $\angle O_3OO_4 > 53°$.

Proof. Suppose $\angle O_3OO_4$ is a minimum, and that $OO_1 \geqslant OO_2$. Then O_2 will be on C_2 and, by translating O_1O_2 parallel to OO_1, we can assume that C_1 touches C, without violating condition A or affecting O_3, O_4. Thus if O_1O_2 meets C_2 again in $P, OO_1 = 2, OO_2 \geqslant \frac{3}{2}, O_1O_2 = 1, O_2P = 1$, and so $OP > \frac{3}{2}$. Hence we can displace O_2 perpendicularly to O_1O_2 till C_2 touches C and then rotate O_2 about O till O_2 is on C_1, without violating condition A, that is, we take $OO_1 = OO_2 = 2,\ O_1O_2 = 1$.

We can show as in the previous lemma that $O_3O_4 = 1$, and that if $1 < OO_3$, $OO_4 < \lambda$, where $\lambda = \frac{3}{2}$ in (1) and $\lambda = 1{\cdot}2$ in (2), then O_3, O_4 lie on C_1, C_2. Then

$$O_1O_2 = O_3O_4 = O_1O_3 = O_2O_4 = 1, \quad OO_1 = OO_2 = 2,$$

and the only variation still possible is a translation of O_3O_4 in its own length. If now $OO_3 \leqslant OO_4$, $\angle O_3OO_4$ decreases as OO_3 decreases and OO_4 increases, and so, in (1), $\angle O_3OO_4$ is not less than its value when $OO_3 = 1$, that is,

$$\angle O_3OO_4 \geqslant \tfrac{1}{2}(180° - \cos^{-1}\tfrac{1}{4}) > 52\tfrac{1}{7}°,$$

for $1 \leqslant OO_3, OO_4 < \frac{3}{2}$, and if $OO_4 = \frac{3}{2}$,

$$\angle O_3OO_4 > 2\ .\ 28\tfrac{3}{4}° > 56°.$$

In (2), $\angle O_3OO_4$ is not less than its value when $OO_4 = 1{\cdot}2$, $1 < OO_3, OO_4 \leqslant 1{\cdot}2$, while if $OO_3 = 1$, then $OO_4 > 1{\cdot}2$, and so

$$\cos O_3OO_4 < (0{\cdot}44 + OO_3^2)/2{\cdot}4OO_3 < 1{\cdot}44/2{\cdot}4$$

and

$$\angle O_3OO_4 > 53°.$$

Lemma 6. *Two unit circles C_1, C_2 satisfying the condition A meet in P, P'. If* min $(OO_1, OO_2) \geqslant \frac{3}{2}$ *and* $\angle O_1OO_2 < 45°$, *then* $OP' > \frac{3}{2}$ *and* $OP < 1{\cdot}2$.

Proof. OP will be greatest when $OO_1 = OO_2 = 2$, and then

$$OP = 2\cos 22\tfrac{1}{2}° - \sqrt{(4\cos^2 22\tfrac{1}{2}° - 3)} < 1{\cdot}2.$$

OP' will be least when $OO_1 = OO_2 = \frac{3}{2}$, and then

$$OP' = \tfrac{3}{2}\{\cos 22\tfrac{1}{2}° + \sqrt{(\cos^2 22\tfrac{1}{2}° - \tfrac{5}{9})}\} > \tfrac{3}{2}.$$

Corollary. Under the conditions of the lemma there is no circle C_3 such that C_1, C_2, C_3 satisfy A, and $1{\cdot}5 \geqslant OO_3 \geqslant 1{\cdot}2$.

Proof of Theorem I. By lemma 1 it is sufficient to prove the result for unit circles. In any configuration satisfying condition A we denote the set of circles C_i, $O_iO \geqslant \frac{3}{2}$, by M and the set of circles C_j, $O_jO < \frac{3}{2}$ by N. Let α be the number of pairs of N with one M-circle between them, and β the number of pairs with two M-circles between them, and so on. Suppose that

there exists a configuration of 19 unit circles satisfying A. We consider the following three cases :

(1) M has 12 members and N has 7 ;
(2) M has 11 members and N has 8 ;
(3) M has 10 members and N has 9.

By the corollaries to lemmas 2, 3, these three cases exhaust the possibilities.

Case 1. If C_i, C_j are two consecutive members of M, then by lemma 2 $\angle O_iOO_j > 28\frac{3}{4}^\circ$, and so $\angle O_iOO_j < 360^\circ - 11 \,.\, 28\frac{3}{4}^\circ < 45^\circ$. Thus by lemma 6 there is no N-circle C_k, $OO_k > 1{\cdot}2$. Hence the angle between two N-circles is greater than $49\frac{1}{7}^\circ$ (lemma 3). Then the sum S of the angular separations of consecutive N-circles is such that (lemmas 2, 3, 5)

$$S > 7 \,.\, 49\tfrac{1}{7}^\circ + \alpha \,.\, O + \beta(53^\circ - 49\tfrac{1}{7}^\circ) + \gamma(57\tfrac{1}{2}^\circ - 49\tfrac{1}{7}^\circ) + \delta(86\tfrac{1}{4}^\circ - 49\tfrac{1}{7}^\circ),$$
$$\alpha + 2\beta + 3\gamma + 4\delta = 12, \quad \alpha + \beta + \gamma + \delta \leqslant 7.$$

Thus $S > 360^\circ - 16^\circ + 3\frac{6}{7}\beta + 8\frac{1}{4}\gamma + 36\delta > 360^\circ$, which is impossible.

Case 2. For the sum S of the angular separations of consecutive N-circles we have

$$S > 310^\circ + \alpha(41\tfrac{1}{3}^\circ - 38\tfrac{3}{4}^\circ) + \beta(52\tfrac{1}{7}^\circ - 38\tfrac{3}{4}^\circ) + \gamma(57\tfrac{1}{2}^\circ - 38\tfrac{3}{4}^\circ),$$
$$\alpha + 2\beta + 3\gamma = 11, \quad \alpha + \beta + \gamma \leqslant 8.$$

Thus $S > 360^\circ$ unless $\alpha = 6$, $\beta = 1$, $\gamma = 1$, when $S = 357\frac{1}{2}^\circ$, and so the angle between two N-circles is less than $2\frac{1}{2}^\circ$ plus the relevant minimum used above. In the case of the six pairs of N-circles enclosing one M-circle this is less than $44^\circ < 49\frac{1}{7}^\circ$. If C_1C_2 is one such pair, then by lemma 3 max $(OO_1, OO_2) > 1{\cdot}2$, say $OO_2 > 1{\cdot}2$, and so if C_3, C_4 are the M-circles enclosing C_2, $\angle O_3OO_4 > 45^\circ$ by lemma 6. There are at least three such pairs of M-circles, and so if S' is the sum of the angular separations of the M-circles, $S' > 3 \,.\, 45^\circ + 8 \,.\, 28\frac{3}{4}^\circ = 365^\circ$, which is impossible.

Case 3. The total angular separation S of the N-circles is such that, (by lemmas 2, 3, 4, 5),

$$S > 348\tfrac{3}{4}^\circ + \alpha \,.\, 2\tfrac{7}{12}^\circ + \beta \,.\, 13\tfrac{1}{4}^\circ + \gamma \,.\, 18\tfrac{3}{4}^\circ,$$
$$\alpha + 2\beta + 3\gamma = 10, \quad \alpha + \beta + \gamma \leqslant 9.$$

Hence $S > 360^\circ$, which is impossible.

Also by lemma 2 we can draw 12 unit circles touching C and 6 unit circles with their centres on C. Thus the theorem is proved. E. R. R.

JOHN WALTER BROOKS.

JOHN WALTER BROOKS, B.Sc., Honorary Secretary of the North-East Branch of the Mathematical Association, died very suddenly on 2nd November, 1948. He had been associated with the Branch since its inception in 1928, and for all but about two years since then had served it in the capacity of Secretary. His life was marked by qualities of sincerity and integrity, which were evident in all his activities.

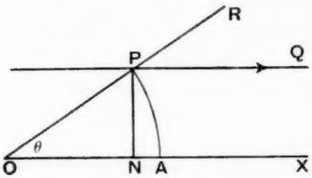
John Brooks was born in 1894, and was educated at Rutherford College School, Newcastle, and at Liverpool University, where he graduated with second class honours in mathematics. He served during the first world war as an infantry officer, and in later years in the Army Cadet Forces and the Air Training Corps.

After teaching for a few years in Newcastle elementary schools, he was appointed mathematics master at Westoe Secondary School, and later at South Shields High School. He was a fine teacher, and was greatly loved and respected by boys and colleagues. J. H.

MATHEMATICAL NOTES.

2029. *Note on the limit of* $(\sin \theta)/\theta$ *when* $\theta \to 0$.

With reference to the article, " A Lesson on π and on $\lim\limits_{\theta\to 0} (\sin \theta)\theta$ " in the *Gazette*, December 1946, p. 282, the late Dr. J. Prescott suggested the following method of presenting the above limit.



Let θ be the angle XOR. With centre O describe a circular arc AP, cutting OX in A and OR in P. Let PN be the perpendicular from P to OX. Then

$$\frac{\sin \theta}{\theta} = \frac{NP}{OP} \Big/ \frac{\text{arc } AP}{OP} = \frac{NP}{\text{arc } AP}.$$

Now draw PQ parallel to OX and let P move " to infinity " along PQ (so that NP remains fixed in length); then, plainly, the limit of the length of the arc AP is the length of NP, so that $NP/\text{arc } AP \to 1$. Also, $\theta \to 0$, and hence

$$(\sin \theta)/\theta \to 1 \text{ when } \theta \to 0.$$

F. B.

2030. *A practical use of continued fractions.*

1. In a marine reduction gear, the drive is split into two trains, as shown diagramatically in Fig. 1. In order to ensure that the load is shared equally

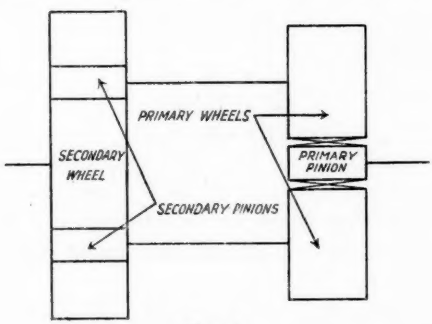


FIG. 1.

by the two trains, it is necessary to adjust the angular position of wheel A by small amounts relative to pinion B.

If A and B are turned together through a whole number of B's tooth pitches, then A will be rotated through a whole number of its tooth pitches, plus a fraction of a tooth pitch, provided that the numbers of teeth on A and B are

prime to each other. As far as the meshing of its teeth is concerned, the whole number may be neglected, and to all intents and purposes the wheel A will have been rotated through the fraction of its tooth pitch.

2. In what follows, a tooth pitch of the pinion B will be referred to as a pinion pitch, p.p. for short, while the tooth pitch of the wheel A will be referred to as a wheel pitch, w.p. for short.

Let w be the number of teeth on the wheel, and p the number of teeth on the pinion.

Suppose A and B are turned together through a pps.

$$p \text{ pps} = w \text{ wps}.$$

Thus,

$$a \text{ pps} = aw/p \text{ wps}$$
$$= m + f/p,$$

where m and f are integers.

The least possible value of f is ± 1, and when this is so, the smallest possible adjustment of $1/p$ of a wheel pitch or $1/pw$ of a revolution is obtained.

a must therefore be chosen so that

$$\frac{aw}{p} = m \pm \frac{1}{p},$$

i.e.

$$aw - mp = \pm 1.$$

This is best done by expressing p/w (or w/p, whichever is less than 1) as a continued fraction, the successive convergents to which are:

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots \frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n},$$

and using the fact that $p_{n-1}q_n - q_{n-1}p_n = (-1)^n$.

In one design A has 157 teeth and B 53 teeth:

$$\frac{53}{157} = \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{25+}\,\frac{1}{2}.$$

Successive convergents are:

$$\frac{1}{2}, \frac{1}{3}, \frac{26}{77}, \frac{53}{157}$$

$$26 \cdot 157 - 77 \cdot 53 = 1,$$

$$26 \cdot \frac{157}{53} = 77 + \frac{1}{53}.$$

Thus a rotation of 26 pps rotates the wheel through 77 wps plus 1/53rd of a wheel pitch, giving an effective adjustment of 1/53rd of a wp or $\frac{1}{53 \cdot 157}$ of a revolution.

3. In practice a finer adjustment is required. The shaft which drives B is driven from A via a toothed coupling as shown in Fig. 2, and a finer adjustment can be obtained by using the coupling teeth.

Let the number of teeth in the tooth coupling be t.

Suppose A and B are together turned through a complete number of pinion pitches a, and then the coupling is split and wheel A turned back through b complete coupling pitches (cps for short). The net rotation of A is obtained as follows:

$$a \text{ pps} = m + \frac{f}{p}$$

$$b \text{ cps} = n + \frac{g}{t}$$

$$a \text{ pps} - b \text{ cps} = \text{whole number} + \left(\frac{ft - gp}{pt}\right)$$

The effective rotation of A is $(ft - gp)/pt$ wps, and the finest possible adjustment is obtained when $ft - gp = \pm 1$, and the amount of this adjustment is $1/pt$ or $1/ptw$ of a revolution (cf. $1/wp$ by the first method).

a and b must be chosen to make $ft - gp = \pm 1$.

f and g can be found by expressing t/p as a continued fraction. Once f and g have been found, it is a simple matter to find a and b.

In the design referred to, $t = 43$,

$$\frac{t}{p} = \frac{43}{53} = \frac{1}{1+}\ \frac{1}{4+}\ \frac{1}{3+}\ \frac{1}{3}.$$

Successive convergents : $\frac{1}{1}, \frac{4}{5}, \frac{13}{16}, \frac{43}{53},$

so that $f = -16$ and $g = -13$.

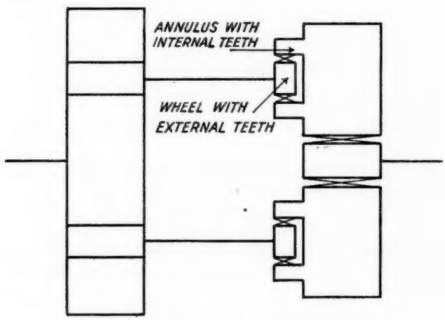


Fig. 2.

It has been shown that a rotation of 26 pps produces an effective adjustment of 1/53rd wp. In a similar way it may be shown that a rotation of 20 cps produces an adjustment of 1/43rd wp. Therefore, a backward rotation of 16×26 pps must be followed by a forward rotation of 13×20 cps.

$$16 \times 26 \text{ pps} = 416 \text{ pps} = 9 \text{ whole revs.} - 8 \text{ pps},$$
$$13 \times 20 \text{ cps} = 260 \text{ cps} = 6 \text{ whole revs.} + 2 \text{ cps};$$

i.e. to produce the finest possible adjustment of $\frac{1}{43\,.\,53\,.\,157}$ of a revolution, the wheel A is turned forward through 8 pps, and then turned forward through 2 cps.

Check : $$8 \text{ pps} = \frac{8}{53} \times 157 \text{ wps} = \left(24 - \frac{16}{53}\right) \text{ wps}.$$

$$2 \text{ cps} = \frac{2}{43} \times 157 \text{ wps} = \left(7 + \frac{13}{43}\right) \text{ wps}.$$

$$\text{Total rotation} = \left\{31 + \left(\frac{13}{43} - \frac{16}{53}\right)\right\} \text{ wps,}$$

or an adjustment of $\dfrac{1}{43\,.\,53}$ of a wp, or $\dfrac{1}{43\,.\,53\,.\,157}$ of a revolution.

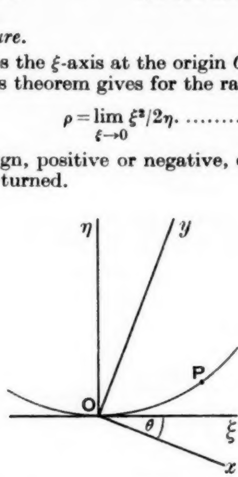
E. P. LOUGHER.

2031. *Radius of curvature.*

If a plane curve touches the ξ-axis at the origin O, and if P is a point, near O, on the curve, Newton's theorem gives for the radius of curvature

$$\rho = \lim_{\xi \to 0} \xi^2/2\eta. \quad \text{(i)}$$

ρ and η have the same sign, positive or negative, depending on the way the concavity of the curve is turned.



Now rotate the axes through the angle θ in the negative sense. Referred to the new axes P becomes (x, y) where

$$\xi = x \cos\theta + y \sin\theta, \quad \eta = y\cos\theta - x\sin\theta,$$

so that ξ, η are functions of x. Thus if dashes denote differentiations with respect to x, two successive applications of l'Hospital's theorem, for the indeterminate form 0/0, give

$$\begin{aligned}\rho &= \lim_{x\to 0} \xi\xi'/\eta' \\ &= \lim_{x\to 0} \{(\xi')^2 + \xi\xi''\}/\eta'' \\ &= \lim_{x\to 0} (\xi')^2/\eta'' \\ &= \lim_{x\to 0} (y'\sin\theta + \cos\theta)^2/y''\cos\theta \\ &= \frac{(1+y_0'^2)^2\cos\theta}{y_0''},\end{aligned}$$

where suffix zero refers to values at $x=0$, so that $y_0' = \tan\theta$.

Since any point on the curve may be taken as origin, we get the usual cartesian formula for ρ at that point.

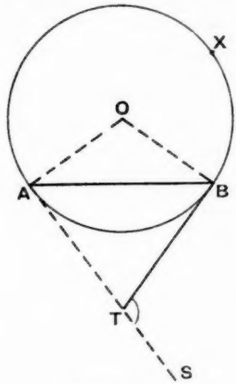
If $\cos\theta$ is retained, however, there is no ambiguity of sign, ρ being positive or negative according as $\cos\theta$ and y_0'' have the same sign or opposite signs. Thus the centre of curvature at (x, y) is

$$\{x - y'(1+y'^2)/y'', \quad y + (1+y'^2)/y''\}.$$

L. M. M.-T.

2032. *The "alternate segment" theorem.*

A short while ago I had occasion to bend some pieces of wood to make an edging for a circular flower bed. While doing so the thought occurred to me that the more flexible the wood the more I could bend it and the less number of pieces I would require to make a circle. Automatically following this line of thought, it next occurred to me that the amount of bending would be measured by the angle between the tangents at the ends, and the fraction of the circumference covered by the wood would be measured by the angle subtended by the wood at the centre. I then realised that I had an unusual proof of the " Alternate Segment " theorem. Here it is :



Given a chord AB and a tangent BT.

Join AO, BO, and draw the tangent ATS at A.

Since the angles OAT, OBT are right angles, $AOBT$ is cyclic.

Hence $\angle BTS = \angle AOB$ at centre.

Since $TA = TB$, $\angle ABT = \frac{1}{2}\angle BTS = \frac{1}{2}\angle AOB$

$= \text{any angle in segment } AXB.$

The two theorems involved—properties of cyclic quadrilaterals and equal tangents—are both usually proved before the alternate segment theorem, so there is no logical objection to the proof, but perhaps the interest lies less in the proof itself than in the manner of its discovery. A. J. M.

2033. *Sign conventions and bending moments.*

Suppose that rectangular axes OX, OY are in the usual directions. Then the positive direction for the measurement of angle is counter-clockwise. In the first place the positive direction could be assigned arbitrarily, but the signs of the circular functions in general use presuppose that the counter-clockwise direction is positive. If a moment or couple is defined by the relation " Couple $= I\alpha$ ", where I and α have their usual meanings, then, as I is positive, couple is measured in the same sense as α, which in turn is measured in the same sense as angle. Hence counter-clockwise couples are positive.

Consider the bending of beams. The fundamental equation is $M/I = E/R$. The second moment of area of the section at P, namely I, is always positive ;

E is equal to stress/strain, and is positive, since the sign of the stress is the same as that of the strain. Hence the sign of the bending moment, M, is the same as that of R, the radius of curvature at P. Since R is approximately equal to the reciprocal of d^2y/dx^2, which is positive in the case illustrated by

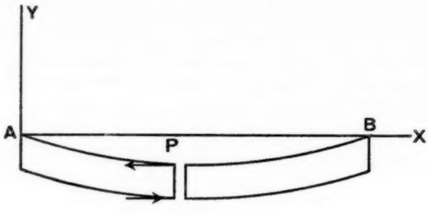


FIG. 1.

Fig. 1, M must be positive in this case. The arrows in Fig. 1 indicate the directions of the forces exerted on AP by PB, and it can be seen that they produce a positive couple. This couple, the bending moment M, can be obtained by taking moments about P of the forces on PB. The case of a beam which is convex up would also lead to the result that " the bending moment at any point is the algebraic sum of the moments about that point of the forces on the positive side of the point ".

It will be noted that, in the case of horizontal beams, this definition of bending moment is equivalent to the engineers' statement that a " sagging " moment (*i.e.* regarding P as fixed, a moment about P which would cause the beam to sag) is positive, and a " hogging " moment is negative. The disadvantage of this latter procedure is that " sagging " and " hogging " have no meaning when applied to vertical columns or struts.

It may be noted that if the positive direction of shear force S is defined as the positive direction of OY, the definitions lead to the usual equations $dM/dx = -S$, $dS/dx = -w$. Here observe that w, the force due to gravity on

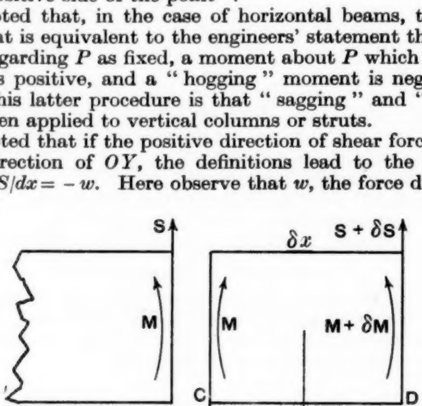


FIG. 2.

unit length of the beam, will be a negative quantity. For, consider the equilibrium of the small part CD of the beam. Taking moments about C,

$$M + \delta M - M + (S + \delta S)\,\delta x - (-w)\,\delta x \,.\, \tfrac{1}{2}\delta x = 0,$$

whence $dM/dx = -S$.

Resolving vertically, $\quad S + \delta S = S + (-w)\,\delta x,$

whence $\quad dS/dx = -w.$

If, as is often the case in practice, the positive direction of OY is downward, we could still adhere to our signs of the circular functions, if the positive sense of angle were clockwise. Moments or couples would be measured in the same sense. The above analysis would still hold with the positive sense of S downward. In all cases, the results can be expressed in the following manner: the positive direction of shear force is the positive direction of OY; the positive direction of bending moment is the direction in which it would be necessary to rotate positive OX so as to reach positive OY, it being understood that shear force and bending moment at a point are due to the forces on the positive side of that point.

An example in which the sign of the bending moment is of fundamental importance is furnished by the cases of laterally loaded struts and tie-bars.

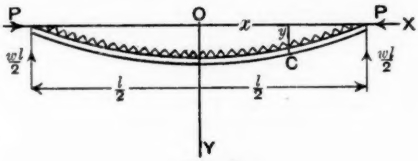


FIG. 3.

In the case of the strut, with axes as shown in Fig. 3, the bending moment at C is given by

$$M = -Py - \tfrac{1}{2}wl(\tfrac{1}{2}l - x) + \tfrac{1}{2}w(\tfrac{1}{2}l - x)^2,$$

so that since $M = EI/R$, we have

$$EI(d^2y/dx^2) + Py = -\tfrac{1}{2}wl(\tfrac{1}{2}l - x) + \tfrac{1}{2}w(\tfrac{1}{2}l - x)^2,$$

a differential equation whose complementary function is composed of circular functions.

In the case of the tie-bar, the direction of P is reversed and hence the sign of its moment is reversed. We have

$$EI(d^2y/dx^2) - Py = -\tfrac{1}{2}wl(\tfrac{1}{2}l - x) + \tfrac{1}{2}w(\tfrac{1}{2}l - x)^2,$$

a differential equation whose complementary function is composed of hyperbolic functions.

M. HUTTON.

2034. *Delirium tremendous.*

(Raved after an attack of H.C. scripts.)

To prove that

$$\int_0^1 x^2\,dx = \tfrac{1}{3}.$$

Try the substitution

$$x = \sin\theta; \quad \therefore\ dx = \cos\theta\,d\theta.$$

Call the integral u.

$$\therefore\ u = \int \sin^2\theta\cos\theta\,d\theta = \int (1 - \cos^2\theta)\cos\theta\,d\theta$$

$$= \int \cos\theta\,d\theta - \int \cos^3\theta\,d\theta.$$

Now
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

~~(Time up.)~~

It is $\therefore$ obvious that

$$u = \tfrac{1}{4}\int \cos\theta\, d\theta - \tfrac{1}{4}\int \cos 3\theta\, d\theta.$$

Now put

$$I = \int \cos\theta\, d\theta.$$

Try the substitution $t = \tan\frac{1}{2}\theta$.

$$\begin{aligned} \therefore\ I &= \int \frac{1-t^2}{1+t^2} \cdot \frac{2dt}{1+t^2} \\ &= \int \frac{2(1-t^2)\,dt}{(1+t^2)^2} \\ &= \int \frac{-2(1+t^2)+4}{(1+t^2)^2}\, dt \\ &= 4\int \frac{dt}{(1+t^2)^2} - 2\int \frac{dt}{1+t^2} \\ &= 4\int \frac{dt}{(1+t^2)^2} - 2\tan^{-1} t. \end{aligned}$$

~~(Time up.)~~

If we write $t = \tan\theta$;

$$\begin{aligned} \therefore\ 4\int \frac{dt}{(1+t^2)^2} &= 4\int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = 4\int \cos^2\theta\, d\theta \\ &= 2\int (1+\cos 2\theta)\, d\theta \\ &= 2\theta + \sin 2\theta \\ &= 2\tan^{-1} t + \sin 2(\tan^{-1} t). \end{aligned}$$

$$\begin{aligned} \therefore\ I &= 2\tan^{-1} t + \sin 2(\tan^{-1} t) - 2\tan^{-1} t \\ &= \sin 2(\tan^{-1} t). \end{aligned}$$

Now the limits of x are 0 and 1.

$\therefore$ The limits of θ are 0 and $\pi/2$.

(I have used θ in two different ways; the second θ should be ϕ, but I haven't time to change.)

$\therefore$ The limits of t are 0 and 1.

When $t = 0$, $\tan^{-1} t = 0$; $\therefore\ \sin 2(\tan^{-1} t) = 0$.

When $t = 1$, $\tan^{-1} t = \pi/4$; $\therefore\ \sin 2(\tan^{-1} t) = 1$.

$\therefore\ I = 1$.

Now let $J = \int \cos 3\theta\, d\theta$.

~~(Time up.)~~

Write $$K=\int \sin 3\theta \, d\theta \, ;$$

$$\therefore \; J+iK=\int e^{3i\theta}\, d\theta$$

$$=\frac{1}{3i}e^{3i\theta}$$

$$=\frac{1}{3i}(\cos 3\theta+i\sin 3\theta).$$

Taking real and imaginery parts,

$$J=\tfrac{1}{3}\sin 3\theta.$$

Using the limits we had before, we get

$$J=\tfrac{1}{3}\sin(\tfrac{3}{2}\pi)$$

$$=-\tfrac{1}{3}.$$

$$\therefore \; u=\tfrac{1}{4}I-\tfrac{1}{4}J.$$

(Time up. I should take the value I found for I and the value I found for J and subtract them and then divide by 4. I haven't time to check the answer, but it ought to follow unless there's a slip in my working.) E. A. M.

2035. *Graphical methods as applied to problems in kinematics.*

Certain pitfalls in applying graphical methods are well known, but these are sometimes so concealed by the wording of a question that even the experienced may fall into them. When a result is to be obtained by the intersection of two graphs, choice of scales do not affect the result. The same is true of area questions, provided ordinates are measured on their own scale and the final area is found by multiplying the base-line, on its own scale, by the mean of the ordinates.

But questions involving gradients are on another footing. The gradient is said to be the tangent of the angle of slope. So it is if the x and y scales are the same. If they are not the same, dy/dx cannot be found by measuring the angle of slope and then finding its tangent. The scaled angle is not the real angle of slope ; but the gradient can be found by measuring its tangent directly, taking ordinate-difference and abscissa-difference each on its own scale.

The following question was recently set in an examination :

" Show that the subnormal for any point on a velocity-space graph gives the acceleration at that point.

The distance in miles and the speed in miles-per-hour of a train are given by the following table :

Distance - -	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2
Speed - -	0	10	28	41	50	56	60	62	63

Find graphically the acceleration when the speed is 50 m.p.h."

From the wording of the question the candidate is led to draw the graph and then to measure NG. Suppose it were possible to take 1 unit along the x-axis for 1 mile and 1 unit along the y-axis for 1 mile-per-hour, this would yield the correct result. But this is quite an unpractical scale. The candidate may very well choose 1 inch to $\frac{1}{4}$ mile along the x-axis and 1 inch to 20 m.p.h.

along the y-axis. But now the geometrical subnormal will give nothing like the right result.

The angle of slope at P is approximately $\tan^{-1} \frac{3}{4}$.

Thus, allowing for the scales, dv/ds is $3 \times 20/(4 \times \frac{1}{2}) = 30$, the units being 1/1 hour.

The acceleration, given by $v\,dv/ds$, is thus 1500 miles-per-hour^2. But NG measured along the x-axis is $1\frac{7}{8}$ in. Interpreting this as $\frac{15}{16}$ miles, consideration of units show that this cannot give an acceleration. If we use the " correct " value of $\tan NPG$, *i.e.* 30 1/hour as above, the corrected value of NG is $30 \times 50 = 1500$ miles per hour^2, as it should be.

It is surely misleading to candidates of the age of those taking the examination to have it suggested, without any qualification, that the " subnormal " —a purely geometrical conception—" gives the acceleration at the point ", especially as it is simpler to make the calculation more simply by the product of v and dv/ds. H. E. P.

2036. *Convergence of series and integrals.*

If the variation of $F(x)$ in the interval $(1, \infty)$ is bounded, the convergence of either $\int_1^\infty F(x)\,dx$ or $\sum_1^\infty F(r)$ entails the convergence of the other. The case where $F(x)$ is positive and decreasing is familiar, but I do not remember to have seen this extension.

We can put $F(x) = f(x) - g(x)$, where f and g are positive decreasing functions. Each of them tends to a limit as $x \to \infty$, and as $F(x)$ must tend to zero, these two limits must be the same.

Now, clearly, $f(r) > \int_r^{r+1} f(x)\,dx > f(r+1)$, and hence, writing

$$S(f) = \sum_n^N f(r), \quad I(f) = \int_n^N f(x)\,dx, \quad (N > n),$$

we have
$$S(f) - f(N) > I(f) > S(f) - f(n).$$

From these and the corresponding inequalities for $g(x)$, we get

$$S(F) - f(N) + g(n) > I(F) > S(F) - f(n) + g(N),$$

and hence the difference between $S(F)$ and $I(F)$ can be made arbitrarily small by increasing n, which proves the proposition.

If we put $F(a+bx) = H(x)$, we see that the series and integral

$$\sum_1^\infty H(r) = \sum_{r=1}^\infty F(a+br) \quad \text{and} \quad \int_1^\infty H(x)\,dx = b^{-1}\int_{a+b}^\infty F(y)\,dy,$$

(where we have put $y = a + bx$), are both convergent or both divergent.

An example is $\qquad F(x) = x^{-h} \sin(x^k), \; (h > k > 0).$

For if we put $x^k = X, \qquad F(x) = X^{-h/k} \sin X = G(X)$, say.

Now in the interval from $n\pi$ to $(n+1)\pi$, $G(X)$ passes from zero to a single maximum or minimum and back to zero, so that its variation is double its greatest numerical value, and is therefore less than $2(n\pi)^{-h/k}$, and the total variation will be less than the sum of a given convergent series. It is easy to see that the integral $\int_1^\infty F(x)\,dx$, and therefore also the series $\sum_1^\infty F(r)$, will converge if $h + k > 1$, diverge if $h + k < 1$, and oscillate between values whose difference is 2 if $h + k = 1$. M. F. Egan.

REVIEWS.

The Collected Works of J. Willard Gibbs. Vol. I, Thermodynamics. Vol. II, Part 1, Elementary Principles in Statistical Mechanics; Part 2, Dynamics, Vector Analysis and Multiple Algebra, Electromagnetic Theory of Light, etc. Pp. xxviii, 434; xviii, 207, vi, 284. $8. 1948 (rep.). (Yale University Press; Geoffrey Cumberlege, London)

The sonorous recitation of the titles of Willard Gibbs' fundamental contributions to mathematical science gives the hearer something of the same sensations as does the recitation of the Duke of Wellington's titles and honours in the concluding chapter of Philip Guedalla's *Life* of that general. But who would not exchange the ephemeral titles of the warrior for the far more lasting *titles* of the works of the mathematician? As the Editor of the *Gazette* has suggested to me, it would be as silly to attempt to "review" these two reprinted volumes as to "review" *Euclid* or the *Principia*. But it is worth while to remind ourselves of how very much alive are almost all of Gibbs' ideas in the world of mathematics to-day.

It is not irrelevant to recall first the story told about Gibbs in Chapter III of Lewis and Randall's *Thermodynamics*. During his long membership in the Yale faculty, Willard Gibbs made but one speech. After listening to a prolonged discussion of the relative merits of language and mathematics as elementary disciplines, he rose to remark "Mathematics is a language", and sat down. In truth Gibbs must have been tremendously interested in language. The great memoir "On the Equilibrium of Heterogeneous Substances" published in the *Transactions* of the Connecticut Academy in 1876 and 1878, where it occupied some 320 pages, gives the reader the same impression as an Act of Parliament: each sentence almost recapitulates everything which precedes it (just as Bohr when lecturing in the early days of the quantum theory used to recapitulate the whole theory each time before he was really happy); and a consequence of this is that not only is each addition to the argument absolutely correct, but its complete generality is provided for by a running undertone of the necessary provisos. Only a master of language can carry through such a programme successfully to such an immense length. Then again, Gibbs' *Vector Analysis* is the invention of a language—a language suitable above all for dynamics.

In the three fields in which Gibbs made supreme contributions—physical chemistry, statistical mechanics, vector analysis and multiple algebras—his notions and his discoveries are still of everyday application. It is now an old story, of how Van't Hoff and other physical chemists re-discovered piecemeal the conditions of dissociative equilibrium and the way any such state altered on changes of pressure or temperature, all of which had been treated with the utmost generality and rigour by Willard Gibbs, though abstractly, with scarcely an example to ease the reader's path; of how the physical chemists used the Van't Hoff equilibrium box to do what Gibbs did by the differential calculus; and of how the relations derived by Nernst from the beautiful theory of the Third Law of Thermodynamics all fitted perfectly into the framework of Gibbs' *Heterogeneous Substances*. The "chemical constant" of an element, the activity or fugacity of a phase, not to speak of the phase rule itself, are all part of the day-to-day equipment of the working physical chemist, and they all involve simple translations out of Gibbs' idiom into forms suitable to modern experimentation. It is indeed extraordinary how Gibbs' thermodynamic calculus *foresaw* phenomena then undiscovered.

Then again, Gibbs' statistical mechanics is as fresh to-day as when pro-

duced; his micro-canonical ensemble and grand ensemble, those dynamical models of thermodynamic systems containing large numbers of unobservable coordinates, have been at once adapted, by Fowler and others, to the needs of the quantum theory. Lastly, however one tries to improve the technique of vector analysis or discover fresh theorems in it, one always finds the result somewhere or other in Gibbs.

The present (1948) reprint is an exact reproduction of the 1928 edition, complete with its introductory note, the preface to the 1906 edition, Bumstead's biographical sketch and the very fine portrait of Gibbs as frontispiece. It is to be most warmly welcomed. E. A. MILNE.

Theory of Probability. By HAROLD JEFFREYS. Second Edition. Pp. vii, 411. 30s. 1948. (Oxford University Press)

Mathematicians who learnt their theory of probability from textbooks of algebra or Whitworth's *Choice and Chance*, and those who have followed up with a reading of the work of modern probabilistes (the French have a word for them) will find little that is familiar in Professor Jeffreys' book. Jeffreys, in fact, is an applied mathematician in the best sense; he is not interested in doing pure mathematics disguised in an "applied" language, but in developing an exact theory of induction with the help of mathematics. The present book is an extension and continuation of the author's *Scientific Inference*, and is one of the most remarkable works on probability since Laplace's *Théorie Analytique*, a masterly piece of constructive scientific thinking which no student of the theory of probability, or of scientific thought in general, can afford to omit from his reading.

Jeffreys is perhaps the principal exponent of the approach which takes probability as an undefined idea obeying certain rules or subjected to certain conventions. For him it is a *theorem* that probability can be expressed as a real number—definitions in terms of proportions of favourable cases or of limiting properties in sequences of events he regards as inadequate for the formulation of a general theory and, indeed, as unsatisfactory in themselves. His theory of induction is based on an extensive use of Bayes' theorem and the so-called "inverse" probability. To make full use of them he has to develop a treatment of prior probabilities which does not, it is necessary to warn the reader, command universal assent. A few years ago this would have been putting it mildly; few scientific topics are subject to such differences of opinion and such heated controversy as inverse probability. But the dust of conflict is settling and there are hopeful signs that the various viewpoints, if not reconcilable, are at least understood. With one or two exceptions they all lead to the same code of conduct in making inferences from statistical data.

This is a standard work, unique of its kind. Even the purest of mathematicians will derive pleasure and profit from it, for it deals with the mathematics of thought itself. M. G. K.

An essay toward a unified theory of special functions. By C. TRUESDELL. Pp. iv, 182. 16s. 1948. *Annals of Mathematics* Studies, 18. (Princeton University Press; Geoffrey Cumberlege, London)

In the study of special functions, Bessel, Legendre, Hermite, Laguerre, we meet with many similarities, with just sufficient differences to be confusing, and to necessitate a reference list ready to hand; such are the differential-difference relations, the expansions of generating functions, integral properties, formulae for nth derivatives, contour integrals. It is easy to suspect that there may be a formal general key to this domain, though the "natural" definitions of the various functions may deter us from supposing that this key, if found, can be more than formal. But if found, it should enable us not only to coor-

dinate known results, but also to give a proof of discovery rather than a mere verification. This is the central idea of Mr. Truesdell's essay, and he himself describes its aim as " to provide a general theory which motivates, discovers, and coordinates . . . seemingly unconnected relations among familiar special functions . . ." ; and again, " It is not my aim to produce a long list of new relations satisfied by various special functions, but rather to render trivial the discovery and proof of a large class of these formulas ". For instance, instead of asking for a verification of the formula

$$e^{-z}L_n(z)=\frac{1}{n!}\int_0^\infty e^{-t}t^nJ_0\{2\sqrt{(zt)}\}dt,$$

he wants to obtain a straightforward general method for attacking the problem in the form " Find an expression for the Laguerre polynomial L_n in terms of Bessel functions ".

The central idea round which he has grouped his material is that coordination can be obtained from a few general theorems concerning the nature of the solutions of the functional equation

$$\frac{\partial}{\partial z}F(z,a)=F(z,a+1).$$

From this, he obtains a generating expansion, an nth derivative formula, an infinite integral, and so on, going far to justify his claim to have answered the questions in which the essay found its origins.

Mr. Truesdell has not attempted to write a clean-cut expository treatise ; the title of " essay " allows him to follow just those paths which he himself has found of interest, but there must be many readers to whom the development will be fascinating in its coordination of so much familiar material. The name of Harry Bateman, in the dedication and the acknowledgments, reminds us of the hope that the encyclopedia of special functions which Bateman was compiling, a task which few others could have attempted, may yet be completed and published. T. A. A. B.

An elementary treatise on differential equations. 2nd edition. By A. Cohen. Pp. vii, 337. 12s. 6d. 1948. (Heath, New York ; Harrap)

This is a second edition, completely revised. The author is Collegiate Professor of Mathematics, The Johns Hopkins University. After a preliminary chapter on the formation of differential equations and the meaning of solutions, we have : II. Differential equations of the first order ; III. Applications ; IV. Additional methods ; V. Singular solutions ; VI. Linear differential equations with constant coefficients ; VII. Linear differential equations of the second order ; VIII. Miscellaneous methods ; IX. Integration in series ; X. Total differential equations ; XI. Systems of simultaneous equations ; XII. Partial differential equations ; XIII. Partial differential equations of the first order ; XIV. Partial differential equations of higher order.

The scope is thus very much that of Piaggio's well-known book. The treatment is rather more concise, the author preferring, as a rule, to state a general problem and its solution before giving examples. This method is suitable to the more mature student ; younger or less able pupils would find the alternative approach, via particular examples to the general case, easier to understand.

Emphasis is placed on the method of substitution with undetermined coefficients as a means of finding a particular integral of a linear differential equation with constant coefficients. Factorisation of the operator is also advocated and the author rightly points out that this method yields the complementary function at the same time. Algebraical manipulation of the

inverse operator is not used, except for resolution into partial fractions; the differential equation having been put in the form

$$y=\frac{a_1}{D-m_1}f(x)+\text{similar terms},$$

the solution is completed by the integration of equations of the form

$$(D-m)y=af(x).$$

Expansion of the inverse operator in series, and the theorem

$$f(D)\{e^{kx}V\}=e^{kx}.f(D+k)V$$

are not used.

In the matter of notation and nomenclature we note that in polar coordinates ρ, θ, ϕ are used where English books use r, θ, ϕ. The author does not like the term "homogeneous" as we use it for ordinary differential equations; the types concerned are included in a wider type, called "isobaric", such that, if $x, y, y', y'', \ldots$ have weights (or dimensions) $1, m, m-1, m-2, \ldots$ all terms of the differential equation have the same weight r; the equation is then solved by the substitutions $x=e^u$, $y=ve^{mu}$.

The book is well written and the examples, which include equations arising in physics, are well chosen. It can be recommended as a sound and concise textbook for fairly able students. C. G. P.

John Couch Adams and the discovery of Neptune. By W. M. SMART. Pp. 56. 5s. 1947. (Royal Astronomical Society, Burlington House, London, W. 1)

Throughout all time Prediction has fascinated mankind. In astronomy, the first glimmerings came with astrology. As far back as 2159 B.C. we are told of Chinese astrologers Hi and Ho being executed for their failure to predict an eclipse. Hence, no doubt, the derivation of that Chinese proverb, "Those that say, don't know, and those that know, don't say". Tycho Brahe, whose quatercentenary was celebrated at the same time as the Neptune centenary, lived so much with the comet of 1577 that he predicted from it the birth of a prince in Finland, who would lay waste Germany and vanish in 1632. Gustavus Adolphus fulfilled this prophecy and died in 1632. *Gulliver's Travels*, published by Jonathan Swift in 1726, contains in the third chapter of the "Voyage to Laputa" the following: (1) Mars has two moons (and in 1877 two moons were discovered in Washington); (2) Swift's highly satirical approximations for the distances and periods of these moons, which evoked from Flammarion the exclamation that Swift had been given "second sight". On the night of 13 March, 1781, William Herschel increased his magnifying power from 227 to 460 and 932 in order to identify what he thought was a comet but which turned out to be the planet Uranus. In 1769, however, Lemonnier had observed it and recorded six observations on a paper bag which had originally contained hair powder. Also Flamsteed, whose tercentenary was celebrated at the same time as the centenary of Neptune, recorded it in 1690. This discovery of Uranus fitted in with a prediction known as Bode's Law, in which the distance of a planet from the sun is proportional to $\cdot4+(\cdot3)2^n$, where $n=1$ for the earth and $n=6$ for Uranus (19·6 against 19·183). Then there are the endless correlations such as "the animal side of man is prominent because the Sun was in Taurus when the human race began". Or again, take a single set of correlations from Halley's comet. The comet of 66 was the sword mentioned by Josephus as hanging over Jerusalem and, according to him, a warning of its impending destruction. The comet of 1066 and all that ushered in the Norman conquest of England. The comet of 1531 saw Henry VIII become head of the church. In 1758 Nelson was born, and the comets of 1835

and 1910 are associated with the figures before and after the general elections, as follows :

	1835	1910
Liberals before election	514	513
Liberals after election	385	397
Opposition before election	144	157
Opposition after election	273	273

What of the next return about 1985? Will the Liberals have to wait until then?

From all these types of prediction, the Neptune discovery rests on an entirely different level.

A. N. Whitehead has told us that the march of human progress is not essentially towards better things, a depressing thought enough, but happily balanced in this field of human endeavour by the march from the realms of vague experience and magic speculations to the exact calculations of astronomical science and the crowning achievement of the Newtonian theory. Laplace hazarded a proposition that a sufficiently developed intelligence, if it were made acquainted with the positions and motions of the atoms at any instant, could predict all future history. The predictions in the *Nautical Almanac* (three years in advance) and the prediction involved in the discovery of Neptune are minor representations of Laplace's dream towards the highest pitch of perfection which the human mind is capable of attaining in the intellectual sphere. No one can view the structure, with the discovery of Neptune as one of its culminating points, without standing in intense admiration for the achievements of mathematicians like Newton, Laplace, Hamilton and company, who have passed through this world on their way, we hope, to higher domains.

It was long before man could distinguish between planets and fixed stars. Even then, as the word πλανητες suggests, they were considered as having erratic rather than regular motions. The motion of a planet is determined in the first place by the mutual attraction which exists between the planet and the sun. By the law of gravitation each planet is also attracted by every other planet and every other body. Modern observations have brought into notice these perturbations of planets due to other bodies. The general case is very complicated and the author heaves a sigh of relief, no doubt, on p. 3 when he says, " happily we need not enter into the intricacies of the subject ". In the solar system, fortunately, the problem of perturbations is simplified by the fact that the perturbing force is small compared with the controlling force, the eccentricities of the elliptical orbits are small and the inclinations of the orbit planes small also generally. Even so, the problems are still amongst the most difficult and the computations involved are most searching. Nevertheless, present practice enables us to determine the positions of the planets at any time within a span of 100,000,000 years plus or minus.

In 1821 A. Bouvard published his tables of Uranus. In 1837 Airy wrote that " the errors of longitude were increasing with fearful rapidity. I cannot conjecture what is the cause of these errors . . . if it be the effect of any unseen body it will be nearly impossible ever to find out its place ". To Bouvard this unknown influence on Uranus crystallised into an unknown planet. On July 3, 1841, John Couch Adams, while an undergraduate at St. John's College, Cambridge, " formed a design of investigating the irregularities in the motion of Uranus, to find out whether they may be attributed to the action of an undiscovered planet beyond it and if possible to determine the elements of its orbit, etc., approximately which would probably lead to its discovery ".

Professor W. M. Smart, in addition to having been John Couch Adams's pro-

fessor of astronomy, has had access to Adams's private papers and has, therefore, set out the whole story in detail from every angle and with many references. The computation of a planet position after allowing for the perturbations of known planets is complicated enough but the inverse problem of analysing the perturbations so as to discover the position of the unknown planet is of much greater difficulty. Adams gave his solution to the latter problem in his paper, " On the perturbations of Uranus " which is published as No. 8 in the *Appendices to Nautical Almanacs* 1834–54 while Airy's paper on " Calculations of Perturbations " is No. 6. It is clear from the outset that Adams is indebted to Pontécoulant's *Théorie Analytique du Système du Monde*, Vol. I, p. 486, for his expression of the perturbations of mean longitude. Meanwhile, Le Verrier published in June 1846 his second memoir announcing the position of the disturbing planet within 1° of that found by Adams. He appealed to Berlin and on September 23, the same evening, Galle took his place at the telescope and called out what he saw to D'Arrest who followed on the map. Neptune was thus found by Galle within 1° of the place predicted by Le Verrier. Professor Smart describes the ensuing controversy about priority in detail. The story makes fascinating reading. One of the main characters, Airy, was undoubtedly a busy man. He was, during this period, called on to attend meetings of the railway gauge committees, to report on the engines of H.M.S. *Janus*, to inspect a new sawmill at Chatham, and to attend meetings of the Tidal Harbour Commission, etc. Challis was unlucky in that he made over 3000 observations without recognising three observations of the new planet. His defect was that he could have started in September 1845 instead of July 1846. Lalande also observed it in 1795, not only on the 10th May but also on the 8th May, and Lamont on October 25, 1845, and September 11 and 17, 1846. Galle, like Adams, was the eldest of seven children, and, although he lived to be 98 years of age, received no special award from a British Society. On pages 50–1 Smart answers Pierce's criticism that the discovery of Neptune was a happy accident.*

An account of the early life of Adams is also given. He is in some ways a model for our schools. He started with the inner urge for a noble science on the practical side and his mathematical training was built up around this practice. He was nurtured in the pure air of the country in the midst of hard work and simple religious piety. He calculated eclipses at the age of sixteen and saw Halley's comet in 1835, and he became Senior Wrangler with 4000 marks while the *proxime accessit* had only 2000 marks.

This volume is worth a place in every school library and everywhere where applications of mathematics are considered. The Royal Astronomical Society is to be congratulated on its new venture. A similar volume could include the talks given to the Forces during the war. The whole is abundant evidence that " the mighty breath which hath invoked my lay " can also descend upon the mathematician and even the technician as well as on the artist, musician, philosopher and poet. A. B.

Nicolaus Copernicus, de Revolutionibus, Preface and Book I. Translated by J. F. Dobson with the assistance of S. Brodetsky. Pp. 32. 3s. 6d. 1947. (Royal Astronomical Society)

The year 1543, in which appeared the epoch-making work of Copernicus, *The Torinensian*, may be regarded as marking the transition from mediaeval to modern science and the *De Revolutionibus* bears, therefore, a message of special significance to all who are concerned with the history of scientific thought.

* *Proceedings of the American Academy of Arts and Sciences*, Vol. I (1846–8).

While the book is remarkable in many ways for its modern outlook and approach, it remains overlaid with much that is characteristically archaic, anthropomorphic and even sentimental. Metaphors such as the following are not uncommon. " In the middle of all sits Sun enthroned. In this most beautiful temple could we place this luminary in any better position . . . ? " Yet there is one celebrated passage in which the idea of universal gravitation " seems to have hovered before the mind of this great man ", and there are many in which strong emphasis is laid upon the essentially relative character of the celestial motions. Thus he argues, "And why not grant that the diurnal rotation is only apparent in the Heavens but real in the Earth? It is but as the saying of Aeneas in *Virgil*—' We sail forth from the harbour, and lands and cities retire.' As the ship floats along in the calm, all external things seem to have the motion that is really that of the ship, while those within the ship feel that they and all its contents are at rest". He explains that " Circular motion ... may be combined with the rectilinear—just as a creature may be at once animal and horse ".

Copernicus sets out to refute the traditional arguments against the heliocentric theory and we find in one place the prescient suggestion : " Let us then leave to the Physicists the question whether the Universe be finite or no, holding only to this that Earth is finite and spherical."

The writing throughout breathes a spirit of humility, reverence and awe. In the preface addressed to Pope Paul III he explains his reluctance to have the volume published on account of the " novelty and incongruity " of his theory ; having " kept it in store not for nine years only, but to a fourth period of nine years " he finally yielded to the persuasion of his friends.

Printed originally as *Occasional Notes* of the Royal Astronomical Society, the pamphlet under review is an example of fine and graceful scholarship. The brief extracts quoted above will serve to convey the quality of the translation from the Latin ; a valuable feature is the annotation which is very full and informative. The excellence of the printing, reproductions and illustrations is outstanding in these days of austerity and the booklet is in every respect up to the high standard of the Society's publications. T. A. Brown.

Surveying instruments, their history and class-room use. By E. R. Keily. Pp. 411. N.p. 1947. Nineteenth Yearbook of the National Council of Teachers of Mathematics. (525 West 120th Street, New York, 27)

" Measurement ", says an Irish poet and writer ", is only a matter of men's prying into the secrets of nature ", and he implies that it is symbolic of " the fall from heaven to earth, which is half the fall of Lucifer ". Brother Edmond R. Kiely, of the Christian Brothers of Ireland, however, invites us to reflect on the romance of measurement. He has had experience in high school and junior engineering mathematics, and from this background he presents a wealth of historical material, which, hitherto, has not been readily available. He then proceeds to adapt the elementary portions to actual mathematical class-room use.

If we could only write the history of science with the same accuracy as the orbit of Neptune, then indeed we should have a science of history. The risings and settings of Venus can be calculated backwards to show us when Amnizaduga lived but the science of history does not rest on such a sure foundation. Much of the story of early invention and development of mechanisms and processes in surveying is necessarily lost beyond recall. Petzval, for instance, produced his prescriptions but his calculations will always be wrapped up in mystery. Nevertheless, this work is an attractive attempt to compress into 237 pages the history of surveying instruments throughout about thirty-two centuries. The Romance of Measurement is here portrayed

for the man who has to work mentally at such a pace that he has no time left to think about the romance. We are told that the earliest Babylonian map tablet dates back to about 2500 B.C. The beginnings of surveying are described, in Egypt, China and Babylonia; then the developments in Greece and Rome are dealt with, next the contributions of medieval Europe, Islam and India, and finally the advances made in Europe during the Renaissance. The instruments discussed are the working tools of a craftsman—the level, plumb rule and right angle or square. Then there are the Astrolabes, Quadrants and Staff combinations, with the emphasis on their use in surveying rather than in astronomy and astrology. Apparently there are about 300 astrolabes in the different museums of Europe, from 2 inch to 2 feet diameter. Special markings were used on Muslim instruments to indicate the times of prayer: if the length of the meridian shadow is x, then the beginning of late afternoon prayer occurs when the length of the shadow is $2x$. Geometric squares and quadrants are also described and on p. 80 an Arabic quadrant is shown. There is a clear picture of Tycho Brahe's quadrant, sighted from the centre and showing the shadow square and the revolution of the vertical plane to give the azimuth. Vernier's adaption of his reading device to the quadrant is described on pp. 176–7 and also in French in the appendix. Despite the fact that Vernier's description is dated 1631, if Bion's catalogue is sufficiently representative, the vernier itself was not much used until the end of the seventeenth century. Jacob's staff, the magnetic compass and the floating compasses of Peregrinus are all described.

Land surveying was undoubtedly practised early in Egypt. Euclid and his brethren were called upon to settle disputes over land boundaries after the Nile had overflowed its banks. Systems of land tenure also developed the polimetrum in 1512, which was the earliest prototype of the theodolite and transit. In 1571, Digges produced a description of his theodelitus with its relationship to the astrolabe. The book was called *Pantometria* and the author, as Aubrey, describes feelingly "cutting glasses in such a particular manner that he could discerne pieces of money a mile off", and again, "These signes are most dangerous for bludde letting, the Moone beyinge in them, Taurus, Gemini, Leo, Virgo, and Capricorne with the last halfe of Libra and Scorpius". Rathborne's book deals with the Theodelite, the Playne Table, Circumferentor, Peractor and Decimal Chayne. Proportional compasses were due to Galileo and early slide rules to Oughtred and Gunter.

Thus, by the end of the seventeenth century all important items of a modern surveyor's equipment were in use, albeit in a crude form. We are left, therefore, to consult other works of reference, such as Stanley's *Surveying instruments*, to show how in succeeding centuries extra refinements have been made possible due to magnificent design and craftsmanship. The theodolite for large surveys to-day is a very different instrument from that of 1571.

The remainder of the book is devoted to the development of practical geometry in schools, and the applications of geometry, and trigonometry to simple surveying. "If the Egyptian scribe, Ahmes, could return for a visit to this earth and witness a modern geometry class in action" what would be his reaction? Would he deplore the abandonment of the art of measurement in geometry and trigonometry? To console the ghost of Ahmes, the author outlines a series of exercises in outdoor measurement. These are skilfully chosen and restricted to the capacities of the immature students. The markings on astrolabes and the uses of quadrants as sundials are barely mentioned, evidently in the determination to make the classroom theory as simple as possible. A wealth of diagrams is supplied, including some from old books, thus making the problems realistic and the general lay-out artistic. Controversy now creeps into the situation. In practical mathematics, many teachers

are averse to the presentation of a logical sequence in geometry, no matter how simple ; so here many teachers will object to the practical work involved. The teacher will certainly have to practise dexterity in the use of his simple equipment, and the usual claim will be made that the whole business wastes too much time. Nevertheless, provided that this work is ancillary and does not interfere with the usual mathematical drill and technique, these ideas are worth a fair trial. It may be, that in the history of mathematics and in a little practical work, some teachers will discover the right outlet for that daily advancement which is certain to be favourably reflected in their pupils.

This volume will supply a cultural need to all surveyors and mathematicians. It should be found in every school library. It is well printed on good paper and its cover, so different from that of the ordinary report, is very attractive.

Looking back over the centuries with the author, an inescapable feeling emerges that measurement, in spite of the Irish poet, is a creation of the human spirit. In the hands of the mathematician, " it has taken man beyond the bounds of earth and helped him to form a fair picture of the infinite ". In the modern world, the eastern detachment of the poet is being challenged by a type of wisdom which is fighting desperately against terrific odds to orient command over matter towards spiritual ends. A. BUXTON.

Guidance pamphlet in mathematics for high school students. Pp. 25. 25 cents ; orders of 10 or more, 10 cents each. Report of the Commission on Post-war Plans of the National Council of Teachers of Mathematics. (*Mathematics Teacher*, 525 West 120th St., New York, 27)

What good is mathematics? The question is frequently asked, sometimes rhetorically with an implied answer, " None " ; but when asked in earnest by the adolescent, it deserves serious attention. This pamphlet is an answer, couched in an informal style which should not frighten the high school pupil for whose guidance it is framed. Its sections deal with mathematics for personal use, trained workers, college preparation, professional workers, women, civil servants, with lists of organisations, graduate schools offering doctorates, and references for further information.

Much of the matter is special to the United States, but there is a good deal of general information which, in this handy collected form, should be useful to pupils and teachers in this country. Those who believe that our American cousins have no standard of value save the dollar should note the statement that " if you enjoy mathematics, take no thought of the vocational uses in the to-morrow—sufficient for this day is the *good* thereof ". T. A. A. B.

La Géométrie et Le Problème de L'Espace. III. L'Édification Axiomatique. By FERDINAND GONSETH. Pp. 108. Frs. 7.50. 1947. (Éditions du Griffon, Neuchatel)

Real geometry, the geometry of the space in which we live, sometimes engages the attention of topologists and differential geometers, but then only when they wish to consider a particular example of the spaces in which they are interested. The algebraic geometer is interested in projective spaces defined over any field, and does not usually care to elaborate the properties of that space, defined over the field of real numbers, in which a particular prime, the " prime at infinity ", is particularised. Consequently, the student who does not study topology or differential geometry very seriously at the university leaves euclidean geometry, which he has surveyed at school with the help of his own intuition and an inadequate set of axioms, passes on to coordinate or algebraic geometry, and may never attain to that knowledge of real space which rewards anyone who studies the consequences which follow from a set of suitably chosen axioms.

In this book Professor Gonseth writes a stimulating essay on the axioms of the real line and the real plane. The pace is slow at first, and quickens when the author begins to build " avec des blocs plus gros et moins exactement taillés ". The divisional headings are : Caractères de la méthode axiomatique ; L'Édification axiomatique de la géométrie élémentaire du plan ; L'Incidence ; L'Ordination sur la droite ; La notion " de même sens " comme notion primitive ; La notion " entre " comme notion primitive ; L'ordination dans le plan ; La congruence sur la droite ; " L'égalité des segments " comme notion primitive ; Le " déplacement sur la droite " comme notion primitive ; La congruence dans le plan ; L'ordre continu ; Le parallélisme ; Conclusion.

As mathematical food for the young this book would, I think, prove more nourishing than the various books on calculus and coordinate geometry which fill our schools. But unless the universities set questions on real geometry in their scholarship papers, books like Professor Gonseth's admirable little work will not be published in this country. D. PEDOE.

A Philosophy of Mathematics. By L. O. KATTSOFF. Pp. ix, 266. $5. 1948. (Iowa State College Press, Ames, Iowa)

Thirty years ago, in his *Introduction to Mathematical Philosophy* Russell characterised the study of the Foundations of Mathematics as a journey of mathematical exploration in the reverse direction. This antithesis between research in mathematics and in the foundations of mathematics is true to-day only in regard to the underlying motives of the work ; in technique the two are no longer distinguishable. The analysis of mathematics " has developed into . . . a new, and it appears to be, most fundamental branch of mathematics ". Like light pulses in an elliptic space, two opposite directions of research are converging and a field of study that once lay on a border line between philosophy and mathematics has, in one of its aspects at least, developed into a mathematical discipline which over-reaches the technical resources of our times.

Kattsoff describes eight formal mathematical systems ranging from Frege's *Grundlagen* to Church's *Conversion Calculus*. This is a very considerable undertaking in a book of less than 260 pages but the attempt to produce an interesting and reliable introduction to these systems is highly successful. There is sufficient detail to give the beginner a concrete picture of the system described without exhausting his interest. References are plentiful and the bibliography extensive ; though there are references to publications bearing a later date the ideas discussed in the book do not go beyond the year 1938.

Hilbert's formalist and Brouwer's intuitionist foundation of mathematics are given a very thorough treatment and the accounts of these systems are illuminated by well-pointed quotations. The formalist-finitist controversy is described with admirable impartiality ; it may be that Kattsoff is mistaken in his estimate of the part played by Gödel's construction of non-demonstrable propositions, for " *P or* not $-P$ " may be true even though P is neither demonstrable nor refutable, but it is certainly true that Gödel's discovery led, at least temporarily, to an abandonment of the formalist programme. A reference at this point to Gentzen's consistency proof, and a comparison of Gödel's and Gentzen's results would certainly be helpful.

Gödel's theorem is given a chapter to itself, but it is doubtful if one unfamiliar with the subject will gain much in his understanding of this great work from the forty-six definitions that Kattsoff lists. For one thing, the reader is given no indication of the importance of definition by recursion or, for that matter, that the functions and attributes defined are recursive.

The main point brought out in the discussion (p. 126) of the formalist-finitist controversy is that a *reductio ad absurdum* proof of the consistency of

the *tertium non datur* with other axioms is a vicious circle argument. This observation applies with equal force to Gentzen's proof of freedom from contradiction.

A Philosophy of Mathematics is written by a philosopher not a mathematician. The chapters on conventional mathematics (Chapters 6 and 13) are written for students of philosophy with little knowledge or experience of mathematics, and lack the definitive outlook and confident certainty of the greater part of the book. These chapters could with great advantage have been omitted. The Cauchy criterion for convergence (p. 69, line 8), for instance, is made nonsensical by the omission of " for all positive p ", and the next sentence on monotonic increasing sequences is meaningless. In the discussion of Dedekind cuts on the following page a section of the set of all rational numbers is said to define a whole number if the lower and upper classes have a greatest and least number respectively, but such a section is not Dedekindean since all the rationals between these least and greatest numbers are not members of either class. On the subject of complex numbers (p. 73) it is said that any number of the form $a+bi$ is a complex number, and $\sqrt{-2}$ is complex because it is of the form $0+i\sqrt{(-2)}$. What is meant, of course, is that $\sqrt{-2}$ is of the form $0+i\sqrt{2}$.

The criticism of Kaufmann (top of page 71) is an error of another kind. We may well talk of approximating to the roots of an equation $f(x)=0$ by a sequence of rationals a_n, if $f(a_n)$ converges to zero, without introducing a real number limit for the sequence a_n. It may be that the term approximating is misleading in this context, but it is quite incorrect to say that the concept of real number is necessarily involved.

On the meaning of numbers Kattsoff accepts the intuitionistic standpoint. " Intuitionists ", he says, " are striking in the right direction when they insist that the fundamental intuition of number is the basis of all mathematics. This is actually an intuition of a property of objects-as-such. Hilbert's insistence upon number as a symbol merely is a mistake ". This view on the meaning of number reflects Kattsoff's general conclusion on the nature of mathematics, which may be summed up in his question : How can mathematics be applied if mathematics is an arbitrary system of signs? " That mathematics has some structural properties in common with reality is evidenced ", Kattsoff maintains, " by every discovery predicted by mathematical physics ". " This is not refuted by arguing that experience might not have verified the prediction. The point is that it did! " (p. 246). " This does not mean that mathematics is an empirical science obtained by the inductive procedures of natural science by means of our senses. But it does mean that mathematics is empirical in the sense that it arises through the observation of ' essences ' ". (p. 247).

This view of the nature of mathematics and the meaning of numbers goes back to Plato ; as Kattsoff gives no indication of the arguments that could be advanced against it, it will perhaps be useful to summarise them here. The case against the Platonic standpoint may be outlined as follows.

The discoveries predicted by mathematical physics tell us nothing about the nature of mathematics in general but only about the structural similarity of some aspect of the physical world to some one branch of mathematics ; to characterise mathematics by an accidental feature of one of its parts is to identify this part of mathematics with the whole.

Success in application may explain our interest in, and may well be the inspiration of, some of the mathematical systems that have hitherto been conceived, but the nature of mathematics is not to be found in its history. The binomial coefficients, for instance, were first discovered in connection with selections from groups of objects, and to this day algebra books exploit

this humble origin in a variety of problems on selections and arrangements, but as mathematical functions the binomial coefficients $\binom{n}{r}$ *derive all their properties* from the identity $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$. We cannot *prove* the binomial theorem by observing that we can select the term $a^{n-r}b^r$ from the product $(a+b)(a+b) \ldots (a+b)$, to n factors, in $n!/r!(n-r)!$ different ways; on the contrary it is the binomial theorem itself that *reveals* the correspondence between the coefficients $\binom{n}{r}$ and the number of selections of r objects from a group of n.

Applied mathematics is possible because the arithmetic of natural numbers is also the arithmetic of common objects, but no change in the physical structure of the world would invalidate the Laws of Arithmetic. To say that number is a property of objects-as-such not only makes an unfounded assertion of the uniqueness of a particular number system, but is a misuse of the term *property*, for an object does not change its essence by changing its colour but a collection changes its identity when its number is changed.

Certainly the formal use of number signs in mathematics is not the only use of number, and to that extent Hilbert was wrong in his conception of number, but the application of numbers in counting and measuring is a logical operation, not an observation of essences.

A pile of stones on the ground is just a pile of stones, but if it is so *used*, it may record the number of sheep in a flock, or a warning of danger, or the idle fancy of the last tramp who passed that way.

Errata

p. 66, line 10, comma omitted between a' and b' and (line 11) between a and b and between a' and b'.

p. 69, line 8, for a read a_i.

p. 81, line 8, for $\sum_{i=1}^{n}$ read $\sum_{i=0}^{n}$.

p. 97, line 11, for " *prvq* " read " *prvp* ".

p. 99, line 5 from end, for X read x.

p. 104, line 19, for $(x)\phi(x)$ read $\hat{x}(\phi x)$.

p. 116, line 2 from end, for " for them " read " form the ".

p. 189, line 12, for the exponent t_r of p_{r+1} read t_1.

p. 215, line 9 from end, for $\frac{4}{10} + \frac{7}{11}$ read $\frac{4}{10}$? $\frac{7}{11}$.

R. L. GOODSTEIN.

Theorie und Anwendung der unendlichen Reihen. Von K. KNOPP. 4th edition. Pp. xii, 583. DM. 39·60. 1947. Grundlehren der mathematischen Wissenschaften, 2. (Springer, Berlin)

Knopp's handbook, one of the best of the excellent *Grundlehren* series, has appeared in a 4th edition, published 1947, of which a copy has just reached us. Of many volumes on infinite series, Bromwich's and Knopp's are outstanding, the former for its stimulating brilliance, the latter for its lucid exposition and careful arrangement.

The well-known English translation of Knopp, published by Blackie, was made from the 2nd German edition, but included a chapter specially written for it on asymptotic series and the Euler-Maclaurin summation formula. This was in its turn incorporated in the 3rd German edition (1931), but one or two other improvements were also effected in the 3rd edition, of which this new edition is substantially a reprint. The most important of these improvements are : (i) a neater proof, due to Andersen, of the Knopp-Schnee theorem on the equivalence of the Cesàro and Hölder summation processes ;

(ii) an addition to Fejér's theorem giving a proof of Weierstrass's theorem on the uniform approximation to a continuous function by a polynomial; (iii) Karamata's proof (1930) of Littlewood's theorem that a series Σa_n in which $a_n = O(1/n)$, if summable by Abel's process, is convergent, much simpler than any earlier proof of this famous and " deep " theorem.

Hardy's theorem that a series Σa_n in which $a_n = O(1/n)$, if summable C_k, is convergent, is now stated without the separate proof of the earlier editions, since it can in fact be inferred from Littlewood's theorem and the known connection between the Cesàro and Abel processes. It may be mentioned here that an extremely simple and elegant proof of Hardy's theorem for a series summable C_1 has been given recently by Zygmund, *Colloquium Mathematicum*, I, 3, p. 225.

T. A. A. B.

The Factorial Analysis of Human Ability. By GODFREY H. THOMSON. Third Edition. Pp. xvi, 392. 20s. 1948. (University of London Press)

An explanation of factorial analysis was given when the first edition of this book was reviewed (*Gazette*, 1940, xxiv, 150). Since then the methods have become much better known, and it may suffice in this review to indicate the major changes that have been made by the author in the second (issued in 1946) and third editions. A section on " Conflict between battery reliability and prediction " has been added to Chapter VI. In Chapter IX, entitled " Sampling error and the theory of two factors ", cuts have been made to allow for a discussion of the estimation of population parameters by means of sample " statistics ", taking account of degrees of freedom, and with special reference to Fisher's z-transformation of the correlation coefficient r. A section on " The number of common factors " has been added to Chapter X, and one on " Oblique factors " to Chapter XI. The last section of Chapter XIV on " Profile correlations " has been replaced by a description of an actual experiment. Chapter XVI on " Simple structure " has been re-written and now becomes three chapters, XVI on " Orthogonal simple structure ", XVIII on " Oblique factors, and criticisms ", and XIX on " Second-order factors ". The old Chapter XIX, appropriately entitled " Stop-Press " at the time, gives way to a new Chapter XXI on " The Maximum Likelihood method of estimating factor loadings " (specially written by D. N. Lawley), and to a final chapter on " Some fundamental questions ". There are four Addenda which did not appear in the first edition, while the mathematical appendix has been revised and added to in consequence of the modernisation of the mathematical statistical treatment, and of recent work done on the subject-matter of the book.

Within the limits of printing difficulties in the way of wholesale revision, a good deal of trouble has been taken to improve the presentation, both in bringing the subject-matter up-to-date and in using more refined statistical methods. The readability is not impaired, and the racy style and undoubted authority of the author make this a book to be commended to those who wish to master a subject which is becoming of increasing importance in the fields of psychology and education. The mathematician will find much to interest him in the exposition of this particular example of a mathematical statistical problem.

J. WISHART.

Elements of Nomography. By R. D. DOUGLASS and D. P. ADAMS. Pp. ix, 209. $3.50. 1947. (McGraw-Hill)

The authors are respectively, Professor of Mathematics and Assistant Professor of Graphics, of the Massachusetts Institute of Technology. This inter-departmental collaboration is to be commended in the writing of tech-

nical books. In the reviewer's experience it is possible, in the Mathematics class-room, to explain the underlying theory of the commoner types of alignment chart to students of an Engineering degree course, and to get them to produce a few passable nomograms on graph paper, in three or four hours. The authors of this book contend—and it must be conceded that their case is proved—that the theoretical solution is but a small part of the task of the professional nomographer. Their purpose is to train the student or engineer to produce an accurately drawn nomogram to specification, that is, for stated ranges of the variables, and this is shown to involve preliminary planning and computation, which is best carried out in a systematic manner.

Thus, the first eight, very short, chapters are devoted to a step-by-step treatment of the problems of uniform and non-uniform scales, scale equations, scale factors and displacements. It is not until Chapter IX (p. 40) that the basic parallel-line diagram for the equation $U+V=W$ is reached. With equal spacing this is called the type I diagram; with unequal spacing, type II. It is shown that by change of variable, this type has very wide application; in fact, the authors assert that probably one-half of all the alignment charts ever made are of type II. There now follow: type III, the N diagram for $UV=W$; types IV and V, using three concurrent straight lines to express the relation $1/U+1/V=1/W$; and type VI, the circular nomogram for $UV=W$. The characteristics, flexibility, and extension of each type by substitution, are discussed and illustrated. Each chapter concludes with exercises for the student. The remaining chapter deals with the representation of equations in four or more variables by compound diagrams built from the standard types and concludes with some thirty pages, somewhat hurried in comparison with the rest of the book, on miscellaneous methods.

The mathematical equipment assumed consists of Algebra to indices and logarithms, and elementary Trigonometry. Coordinate Geometry is not used until the final section; it could with advantage have been used to introduce type V, by transforming the equation $x/a+y/b=1$ into polar coordinates.

Some misprints, not serious, are: p. 118, line 8, sign = missing; p. 139, below fig. 76, + for =; p. 163, fig. 92 (*a*), u for μ; p. 187, line 14, − for =. It is suggested that fig. 85 (*b*) with its explanatory paragraph below fig. 86 are confusing in respect of notation. C. G. P.

Mathematical Table Makers. By R. C. Archibald. Pp. v, 82, with 20 plates. $2. 1948. The *Scripta Mathematica* Studies, No. 3. (*Scripta Mathematica*, Yeshiva University, New York)

This excellently printed and bound volume contains " a revised, re-arranged, and somewhat extended reprint of two articles appearing in *Scripta Mathematica* in 1946, together with three additional sketches and portraits ". There are now articles on fifty-three leading table makers, from Vieta, Stevin, Napier, Bürgi and Kepler down to the present day, with portraits of twenty of them. For each of the fifty-three are given biographical notes; references to the location of paintings, busts, monuments, published reproductions of these, or photographs; selected references to biographical information; and a list of published tables. Some points are treated in considerable detail: five pages, for instance, are devoted to an account of portraits, etc. (authentic or otherwise) of Kepler, ending with the conclusion that there are two, and only two, portraits of Kepler, which may with certainty be regarded as true likenesses. The reviewer has found much of value on consulting the original *Scripta Mathematica* articles on a number of occasions during the last two years, and has no doubt that many will turn with profit and convenience to the bound volume. A. Fletcher.

Mechanics. By J. C. SLATER and N. H. FRANK. Pp. xiii, 297. 24s. 1947. (McGraw-Hill)

Professors Slater and Frank have succeeded in writing a book on Classical Mechanics of absorbing interest. It is the first of a new series intended to replace the authors' *Introduction to Theoretical Physics*, which appeared in 1933, and is intended primarily for the use of theoretical physicists. It can, however, be highly recommended to students reading Mathematics as a complement to the more academic textbooks where the emphasis is often on " examples ".

The book is written in a way certain to appeal to the reader ; one welcomes particularly the general introduction on the Newtonian System, for this, as the authors point out, still underlies the whole of Modern Physics. It is right that the student should be able to read some such account of the Newtonian Scheme, however brief. The vector notation, which is becoming rapidly adopted in this country, is used throughout.

The familiar topics follow ; to give some chapter headings : The linear oscillator. Motion in two and three dimensions. Lagrange's and Hamilton's equations. The motion of a symmetrical rigid body. Coupled systems and normal coordinates. The vibrating string. The vibrating membrane. Stresses, strains, and vibration of an elastic body. Flow of fluids.

There are in addition several appendices on numerical solutions of differential equations, vectors, tensors, Fourier analysis, curvilinear coordinates, Bessel functions. Also numerous examples at the end of each chapter and an appendix, some being of considerable interest in themselves.

If a criticism may be directed against the book, it is that the chapter on the Lagrangian method is not as general as it might have been. The generalised coordinates as such are not introduced, but only the coordinates of the particles of the system, and nowhere do the authors deal with the case of " varying relations ". Moreover, in the writer's opinion the aim and spirit of the Lagrangian method is not fully brought out.

The Lagrangian method, nevertheless, is used in most of the book and an interesting comparison is made, in the case of the motion of a symmetrical body, with Euler's equations. There are few misprints, the only ones noted being obvious ones on pp. 7 and 52. V. C. A. F.

Cours de mécanique rationnelle. II. Dynamique des systèmes matériels. Par J. CHAZY. 3rd edition. Pp. vi, 511. 1100 fr. 1948. (Gauthier-Villars)

The third edition of the first volume of Professor Chazy's book was noticed recently in these columns. The second volume contains thirteen chapters, nine of which are concerned chiefly with rigid dynamics ; the remaining four deal with the equilibrium of strings, hydrostatics, hydrodynamics, and Newtonian attraction and potential. The exposition throughout is lucid and efficient ; by postponing Lagrange's equations until half-way through the volume, the author may occasionally sacrifice elegance, but this is more than compensated for by forcing the student to think about principles instead of imagining that in dynamics all he need do is to " put down some equations and work them out ". The long chapter on motion about a fixed point is admirable, with its clear discussion of the Poinsot movement, and the detailed and well-illustrated account of the various forms of motion of a top. Another long and equally good chapter is that on " Chocs et percussions ". There are no exercises for the reader, but we may recall that Vol. I contains a good selection from French examination papers during the last 20 years.

The chapter on hydrodynamics is only some 30 pages in length, and so contains practically nothing beyond the general equations. But the account

is very clear and might well serve a student as a good introduction to the classical treatises on this topic.

The treatment of the equilibrium of strings suggests an expository point. Is it not better to write down the vector equation of equilibrium for a finite portion of the string, in the integral form, and then infer the local (differential) equations of equilibrium? This process seems to have two advantages: it is closer to reality, and it does not involve a simultaneous grapple with statical and calculus difficulties. That it is the method adopted by de la Vallée Poussin will recommend it to most teachers.

Professor Chazy has written on familiar topics, but his lucidity and freshness make the book well worth reading. T. A. A. B.

Electricity. By C. A. Coulson. Pp. xii, 254. 10s. 6d. 1948. (Oliver & Boyd)

The author's object in this worthy addition to the series of University Mathematical Texts is to give a consistent short mathematical account of electrical and magnetic phenomena for university students. To this end he avoids digressions on physical matters such as electrolysis or applied electricity. The notable point about the book is that the macroscopic phenomena of electricity are throughout related to their atomic origin, and this physical background is surveyed, with a general account of the developments of the subject, in the first chapter.

The other chapter headings are as follows: electrostatics; conductors, dipoles and condensers; dielectrics; steady currents; magnetic effects of currents; steady currents in magnetic material; permanent magnetism; potential problems; special methods; induction; alternating current theory; Maxwell's equations; units and dimensions.

A consequence of the atomic viewpoint adopted is that the theory of magnetism is developed without that fictitious entity, the magnetic pole. By starting from the resemblance of the interaction between currents in two small coils to that between two electric dipoles, the parallelism between the treatment of the two fields is made convincing, in spite of the more artificial appearance of **H**. The extension to permanent magnetism by considering atomic currents, too, is an improvement in treatment.

The general view of the subject is clarified by confining to two chapters the analytical methods used for potential and for two-dimensional problems. Vector notation is used consistently, and the comprehensive chapter on units does well to emphasise the rationalised and m.k.s. units.

Each chapter has an extensive collection of examples, which enable the reader to extend greatly the field covered in the text. R. B. H.

The Strange Story of the Quantum. By Banesh Hoffmann. Pp. xi, 239. $3. 1947. (Harper & Brothers, New York)

The remarkable way in which the conflict between the wave and particle aspects of radiation has been resolved makes a fascinating story but one extremely difficult to make clear even to those with some knowledge of physics. A thoroughgoing attempt to present the subject to the intelligent layman has not been attempted before but Dr. Hoffmann has now succeeded in writing a most exciting and, indeed, absorbing story. Assisted by that absence of literary inhibition which characterises American writing, he has portrayed the conflict of ideas between the wave and particle theories in semi-military terms thereby making the whole story read almost as an epic. It is true that for British readers the style may not appeal to the same extent as to transatlantic readers, and a further and perhaps more important criticism is that the uninitiated may gain the impression that scientific research consists

of a periodic destruction of all that has gone before and replacement by an entirely different set of " facts ", a fallacy all too prevalent among those who have had the misfortune not to have included science in their education. Nevertheless, Dr. Hoffmann has not sacrificed accuracy to make a good story and even one familiar with physics would benefit from reading the book. Many of the analogies introduced to make clear some unusual concept are very well chosen and it is often illuminating even to a specialist to realise the relation of an apparently very theoretical idea to everyday occurrences.

Now that Dr. Hoffmann has shown that even the complicated study of the emergence and triumph of quantum theory can be described in ordinary language it is to be hoped that other authors will attempt similar popular versions of the progress of theoretical physics. It is not too much to hope that eventually a similar service might be done for those branches of pure mathematics which at present are barely appreciated except by a narrow circle of specialists working almost exclusively in one branch.

H. S. W. M.

MATHEMATICAL FILM STRIPS.

Introduction to Area. By D. M. Barrett. (CGB 146) 41 pictures, with teaching notes, pp. 18. 12s. 6d. (Common Ground Ltd.)

This film strip is intended for the primary school. It begins with some practical examples of the measurement of area by counting squares, leading to the choice of the square inch and larger units. This is applied to the calculation of the area of a rectangle and of areas built up from rectangles, which the notes describe as irregular.

The choice of material is good, the notes adequate, and the drawings and photography up to the usual high standard. Teachers in need of additional material at this stage will find this film strip useful. This method of treatment could be applied with advantage to more difficult problems.

Theorem of Pythagoras. By A. M. Khan. (CGB 141) 36 pictures, with teaching notes. Pp. 26. 12s. 6d. 1948. (Common Ground Ltd.)

A formal statement of the theorem is followed by various demonstrations by dissection, the diagrams for eight different proofs, four applications and the usual textbook problems and theoretical exercises.

It is a pleasant surprise to find so much material on a topic which has become stereotyped and almost hackneyed. The first part is of historical interest and should stimulate a class which knows the theorem. Some of the later exercises cling too closely to the textbook, but there is ample compensation for this in the historical detail contained in the notes. The drawing and photography are up to the usual high standard.

Pythagoras and his Theorem. By Marion Ray. 13 pictures, in colour. 15s. (Marion Ray.) 1947.

This is Euclid's time-honoured proof, broken down into a number of diagrams ; these are preceded by two pictures which give a slight historical background. The drawing and photography are excellent, but the choice of material too limited. Even those teachers who still use this proof will find this little more than a diversion.

Le Dessin Animé. By J. L. Nicolet. Pp. 32. Fr. 2.80. (Scientifilm A. Colomb). Lausanne, 1944. **Intuition Mathématique et Dessins Animés.** By J. L. Nicolet. Pp. 30. Fr. 2.00. (Librairie Payot.) Lausanne, 1942.

Mathematical films in England have been confined, in their aims, to the

elucidation of a topic by illustration, or by visual notation, or by fitting it into a " real life " background. In Switzerland M. Nicolet has produced a new type of cartoon film, which he describes in these two booklets. These films aim at the development of mathematical intuition. A series of animated diagrams are shown in such a way as to lead the audience to discover for itself the mathematical result, which is then formally proved on a blackboard.

Without the actual films, which are not available here, it would be idle to attempt an assessment of the method and its claims. The booklets are interesting and obviously the work of an enthusiast ; they should be read by all who are concerned with new ideas and new methods in teaching.

I. R. V.

Mathematics for the Modern School. By I. R. VESSELO.

No. 1. The Football Field. 27 pictures. Teaching notes, pp. 7.
No. 2. The Bicycle. 27 pictures. Teaching notes, pp. 7.
No. 3. Chess. 34 pictures. Teaching notes, pp. 8.

Each 12s. 6d. 1948. (Educational Publicity Ltd.)

The field of mathematics in the modern school is still in a very experimental stage. The work of Mr. Vesselo and his committee in the field of visual aids is well known, and teachers will welcome this contribution.

In the first two strips interest in a game or an everyday object is used to lead to mathematical topics. Thus the football field takes one breathlessly through problems of levelling and area, marking the pitch, testing the lines and shooting at goal (for success at which, it seems, one must know several theorems on angles in circles). The bicycle serves to introduce : a little about the ellipse, gears and gear ratios, tangents to circles, loci, including the epicycloid and hypocycloid, moments and centre of gravity.

The film strips do no more than introduce these topics, but they do so in such an interesting setting that they are, at the least, worth trying. Many teachers will be glad to have the material provided, although they may differ in the way they use it.

The third strip, on chess, is in a different category. This being assumed new, the treatment is much slower. The game is treated as a source of mathematical material, although of a very elementary nature ; as a result there are many digressions.

The drawings and photography are very well done, and the ideas behind them excellent. In the hands of a good teacher they should do much to develop an interest in mathematics, even in those who are not likely to acquire much technical skill.

C. M. H.

No. 300. CORRIGENDA

P. 102, 1–5 : *for* directrix *read* conic.

P. 162, " Editors of the *Gazette* " : *for* Nos. 16–208 W. J. Greenstreet *read* Nos. 12–208 W. J. Greenstreet.

P. 203, l. 20 : *for* $1 \cdot a_1 a_2 \ldots a_{n-1}\alpha$ *read* $1 \cdot a_1 a_2 \ldots a_{n-1}\dot{\alpha}$.
l. 21 : *for* $1/10^n$ *read* $1/10^N$.

P. 190, Fig. : *for* S *read* s.
l. 5 : *for* $\vec{V}s$ *read* $V\vec{s}$.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS GLASGOW

ng it
ced a
These
nated
itself
.
lle to
s are
read

R. V.

Ltd.)
ental
l aids

ed to
dlessly
lines
veral
about
g the

so in
Many
differ

umed
ce of
result

ehind
ch to
equire
M. H.

t *read*